Cyclic behaviour of Volterra composition operators

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Cyclic behaviour of Volterra composition operators

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Abstract

We determine the cyclic behaviour of Volterra composition operators, which are defined as

$$(V_\varphi f)(x) = \int_0^{\varphi(x)} f(t)\,dt, \quad f \in L^p[0,1], \ \ 1 \leqslant p \leqslant \infty,$$

where φ is a measurable self-map of $[0,1]$. The cyclic behaviour of V_φ is essentially determined by the behaviour of the inducing symbol φ at 0 and at 1. As a particular result, we provide new examples of quasinilpotent supercyclic operators, which extend and complement previous ones of Héctor Salas.

1. *Introduction*

For each Lebesgue measurable self-map φ of the unit interval $[0,1]$ and each $1 \leqslant p \leqslant \infty$, the *Volterra composition operator* is defined as

$$(V_\varphi f)(x) = \int_0^{\varphi(x)} f(t)\,dt, \quad f \in L^p[0,1],$$

which is always measurable because it is the composition of an absolutely continuous (difference of increasing functions) function with the measurable function φ. If φ is the identity map, then the operator V_φ is just the classical Volterra operator, which as usual is denoted by V. Observe that $V_\varphi = C_\varphi V$, where C_φ denotes the operator that to each function f assigns the function $f \circ \varphi$. Since C_φ is bounded from $L^\infty[0,1]$ into itself and V is bounded from $L^p[0,1]$ into $L^\infty[0,1]$ is compact (see [**1**, p. 44]), it follows that V_φ acting on $L^p[0,1]$ is compact.

Whitley [**24**] and Tong [**22**] independently (see also [**15**, Corollary 2.2]) proved that V_φ is quasinilpotent if and only if $\varphi(x) \leqslant x$ for each $0 \leqslant x \leqslant 1$. The cyclic behaviour of an operator depends much on the behaviour of its iterates and although there is a formula for the iterates of Volterra composition operators, this formula is not handleable. However, this handicap may be overcome.

In Section 2, we show that, for $\varphi(x) = x^\alpha$, $0 < \alpha \leqslant 1$, the operator V_φ is cyclic, with cyclic vector the non-zero constant functions. Indeed, the constant function 1 is cyclic for V_ψ, with $\psi(x) = 1 - \varphi(1-x)$, if and only if the eigenfunctions of V_φ span $L^2[0,1]$. In particular, there are cyclic Volterra operators with the graph of their symbols under the graph of the identity.

In Section 3, we deal with the asymptotic behaviour of the norms of iterates of V_φ. Indeed, for the most interesting class of symbols, the sequence $\{\|V_\varphi^n\|^{1/n^2}\}$ tends to a quantity that depends only on $\varphi'(0)$ and $\varphi'(1)$.

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In Section 4, in order to obtain positive results on the supercyclicity of V_φ as well as on the hypercyclicity of $I + V_\varphi$, we need to extend Salas's theorem [**19**] on the hypercyclicity of perturbations of the identity by backward weighted shifts. We prove a new criterion for an operator acting on a Fréchet space to be hypercyclic.

In Section 5, we deal with supercyclicity and hypercyclicity of Volterra composition operators. Salas in [**20**] asked whether the classical Volterra operator is supercyclic or not, which was answered in the negative in [**6**] (there is another published paper by León-Saavedra and Piqueras in which they intended to prove the same result; however, their proof contains unfixable and serious errors). Indeed, the Volterra operator is not even weakly supercyclic [**17**]. Thus the fact that there are symbols below the main diagonal that supply supercyclic Volterra composition operators is striking. Indeed, using the results of the previous two sections, we show that, for every strictly increasing continuous φ with $\varphi(x) < x$ for $0 < x \leqslant 1$ (note that $\varphi(1) < 1$), the operator V_φ is supercyclic and the operator $I + V_\varphi$ is hypercyclic. For strictly increasing φ with $\varphi(x) < x$ for $0 < x < 1$, $\varphi(1) = 1$ and analytic at 0 and at 1, it is shown that if $\varphi'(0)\varphi'(1) > 1$, then V_φ is supercyclic and if $\varphi'(0)\varphi'(1) < 1$, then V_φ is not even cyclic. The latter applies to the symbols defined by $\varphi_\alpha(x) = x^\alpha$ with $1 < \alpha < \infty$. We close the section with a complete characterization of hypercyclicity of V_φ on Fréchet spaces, which extends recent results by Herzog and Weber [**11**].

2. *Basic cyclic properties of V_φ and examples*

Recall that an operator T on a Banach space $\mathcal{B}$ is *cyclic* if there is f in $\mathcal{B}$ such that span $\{T^n f : n \geqslant 0\}$ is dense in $\mathcal{B}$. We begin with the parametric family of symbols $\varphi_\alpha(x) = x^\alpha$, which motivates a more thorough study of the symbols φ that induce cyclic Volterra composition operators.

Proposition 2.1. *Assume $\varphi(x) = x^\alpha$ with $\alpha > 0$. Then $\tau(x) = x^\beta$ with $\beta > -1/p$ is cyclic for V_φ acting on $L^p[0,1]$, $1 \leqslant p < \infty$, if and only if $0 < \alpha \leqslant 1$.*

Proof. An elementary computation shows that $(V_\varphi^n \phi)(x) = cx^{\beta\alpha^n + (\alpha - \alpha^{n+1})/(1-\alpha)}$, for each $n \geqslant 0$, where $c \neq 0$ depends on n, α and β and for $\alpha = 1$ the second term in the exponent does not appear. Thus, for each $1 \leqslant p < \infty$, the result follows from the Müntz theorem. □

Proposition 2.1 is even true on the space $\mathcal{C}_0[0,1]$ of continuous functions on $[0,1]$ vanishing at 0, endowed with the supremum norm.

The cyclicity of the constant function 1 for V_φ is also possible when $\varphi(x) < x$ for $0 < x < 1$ (see Theorem 2.2 and Corollaries 2.3 and 2.4).

We use the notation and some results from [**15**]. Associated to V_φ there is a function $\mathcal{F}^\varphi(x,z)$ defined on $[0,1] \times \mathbb{C}$. In [**3**], there is also a function $D_{V_\varphi}(\lambda)$ that plays the role of $\mathcal{F}^\varphi(0,z)$. A detailed exposition of the construction and properties of $\mathcal{F}_x^\varphi(z) = \mathcal{F}^\varphi(x,z)$ can be found in [**15**, § 5].

As usual, let $\mathcal{H}^0_{1/2}(\mathbb{C})$ denote the space of entire functions of order strictly less than 1/2 or of order 1/2 and type 0. We can prove the following theorem.

Theorem 2.2. *Let φ be a continuous self-map of $[0,1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$ and set $\psi(x) = 1 - \varphi(1-x)$. If the span of the generalized eigenvectors of V_φ is dense in $L^2[0,1]$, then the constant function 1 is cyclic for V_ψ. The converse is also true, provided that $\mathcal{F}_0^\varphi$ belongs to $\mathcal{H}^0_{1/2}(\mathbb{C})$.*

Proof. In [**15**, Proposition 5.2] it is proved that the map $x \mapsto \mathcal{F}_x^\varphi$ is continuous from $[0,1]$ into the space of entire functions. Hence, for each non-null h in $L^2[0,1]$, we find that

$$G^h(z) = \langle \mathcal{F}^\varphi_{\varphi(\cdot)}(z), h\rangle = \int_0^1 \mathcal{F}^\varphi_{\varphi(x)}(z)\overline{h(x)}\,dx, \quad z \in \mathbb{C}$$

is an entire function. By [**15**, Proposition 5.2] the Taylor coefficients of G^h are given by

$$G^h_n = (-1)^{n-1}\langle UV_\psi^n 1, h\rangle = (-1)^{n-1}\langle V_\psi^n 1, Uh\rangle, \tag{2.1}$$

where $(Uf)(x) = f(1-x)$.

Proceeding by contradiction, suppose now that the constant function 1 is not cyclic for V_ψ. Then there is a non-zero h in $L^2[0,1]$ such that $\langle V_\psi^n 1, Uh\rangle = 0$ for each $n \geqslant 0$ and, therefore, $G^h = 0$. Thus, since in [**15**, Proposition 5.2] it is proved that

$$\frac{\partial \mathcal{F}^\varphi}{\partial x}(x,z) = z\mathcal{F}^\varphi(\varphi(x), z), \tag{2.2}$$

it follows that

$$\int_0^1 \frac{\partial \mathcal{F}^\varphi}{\partial x}(x,z)\overline{h(x)}\,dx \equiv 0. \tag{2.3}$$

Upon differentiating this with respect to z, we obtain

$$\int_0^1 \frac{\partial^{k+1} \mathcal{F}^\varphi}{\partial x \partial z^k}(x,z)\overline{h(x)}dx \equiv 0 \quad \text{for each } k \geqslant 0.$$

From [**15**, § 5], we know that the generalized eigenfunctions of V_φ belong to

$$\operatorname{span}\left\{\frac{\partial^{k+1} \mathcal{F}^\varphi}{\partial x \partial z^k}(x,z) : k = 0, 1, \dots\right\}. \tag{2.4}$$

Therefore, it follows that h is orthogonal to each generalized eigenfunction of V_φ and, thus, the span of the generalized eigenfunctions is not dense in $L^2[0,1]$, which is a contradiction.

Suppose now that $\mathcal{F}_0^\varphi$ belongs to $\mathcal{H}^0_{1/2}(\mathbb{C})$ and the constant function 1 is cyclic for V_ψ. If the span of the generalized eigenfunctions of V_φ is not dense in $L^2[0,1]$, then there is a non-null function h in $L^2[0,1]$ such that h is orthogonal to each generalized eigenfunction of V_φ. Now, by the definition of G^h, the basis of (2.4) (see [**15**, Theorem 5.7] and (2.2)), we have that each zero of $\mathcal{F}_0^\varphi$ is also a zero of G^h of, at least, the same multiplicity. Hence, $H(z) = G^h(z)/\mathcal{F}_0^\varphi(z)$ is an entire function. In addition, by [**15**, Corollary 5.4] and the monotonicity on x of the maximum modulus of $\mathcal{F}_x^\varphi(z)$, we have that

$$M(G^h, R) \leqslant \int_0^1 M(\mathcal{F}^\varphi_{\varphi(x)}, R)|h(x)|\,dx \leqslant \int_0^1 M(\mathcal{F}^\varphi_0, R)|h(x)|\,dx = M(\mathcal{F}^\varphi_0, R)\|h\|_1.$$

Therefore, G^h is in $\mathcal{H}^0_{1/2}(\mathbb{C})$, and so is H. Again, by [**15**, Corollary 5.4], we find that

$$|G^h(-R)| \leqslant M(G^h, R) \leqslant M(\mathcal{F}^\varphi_0, R)\|h\|_1 = \mathcal{F}^\varphi_0(-R)\|h\|_1.$$

Hence, $|H(z)| \leqslant \|h\|_1$ for each z real and negative. Since H is in $\mathcal{H}^0_{1/2}(\mathbb{C})$, the Phragmen–Lindelöf theorem (see [**14**, Theorem 22, p. 50]) implies that H is constant. Hence $G^h = c\mathcal{F}^\varphi_0$, where c is a constant.

Now, for $0 < x \leqslant 1$ set $\phi(x) = \inf\{t \in [0,1] : \varphi(t) \geqslant \varphi(x)\}$. Since $\varphi(x) \geqslant \phi(x) > 0$ for $0 < x \leqslant 1$, we may apply [**15**, Lemma 5.9], for $\alpha = 0$ and $\beta = \phi(x)$ for each $0 < x \leqslant 1$, to obtain

$$\mathcal{F}^\varphi_0(-R) \geqslant (1 + \phi(x)R)\mathcal{F}^\varphi_{\varphi(x)}(-R) \quad \text{for each } R > 0 \text{ and } 0 < x \leqslant 1.$$

This inequality, along with $c\mathcal{F}_0^\varphi = G^h$, implies, for each $R > 0$, that

$$\begin{aligned}|c|\mathcal{F}_0^\varphi(-R) &\leqslant \int_0^1 \mathcal{F}_{\varphi(x)}^\varphi(-R)|h(x)|\,dx \\ &\leqslant \int_0^1 \frac{\mathcal{F}_0^\varphi(-R)}{1+\phi(x)R}|h(x)|\,dx \\ &\leqslant \mathcal{F}_0^\varphi(-R)\|h\|_2 \left(\int_0^1 \frac{dx}{(1+\phi(x)R)^2}\right)^{1/2}.\end{aligned}$$

Therefore,

$$\frac{|c|}{\|h\|_{L^2}^2} \leqslant \int_0^1 \frac{dx}{(1+\phi(x)R)^2} \quad \text{for each } R > 0.$$

Since the integral above tends to 0 as R tends to ∞, we see that $c = 0$. Thus G^h is identically zero and so are its Taylor coefficients. Consequently, from (2.1), we find that $\langle V_\psi^n 1, Uh\rangle = 0$ for each $n \geqslant 0$. Since Uh is different from 0, the constant function 1 cannot be cyclic for V_ψ, which is a contradiction. □

As a direct consequence of [**15**, Corollary 5.22] and Theorem 2.2, we have the following corollary.

COROLLARY 2.3. *Let φ be a continuous self-map of $[0,1]$ with $\varphi(x) > x$ for $0 < x < 1$ and assume that*

$$\varlimsup_{x\to 0} \frac{\ln(\varphi(x)-x)}{\ln x} < 2 \quad \textit{and} \quad \varlimsup_{x\to 1} \frac{\ln(\varphi(x)-x)}{\ln(1-x)} < 2.$$

Then the constant function 1 is cyclic for V_ψ, where $\psi(x) = 1-\varphi(1-x)$, if and only if the span of the generalized eigenfunctions of V_φ is dense in $L^2[0,1]$.

In particular, the above corollary applies to V_ψ, where $\psi(x) = 1-(1-x)^{1/2}$. The next corollary follows from Theorem 2.2 and [**15**, Corollary 5.23], which ensures, under the same hypotheses on φ in the corollary below, that the order of growth $\rho(\mathcal{F}_0^\varphi) = 0$. Therefore, we have the following corollary.

COROLLARY 2.4. *Let φ be a continuous self-map of $[0,1]$ with $\varphi(x) > x$ for $0 < x < 1$. Assume also that φ is differentiable at 0 and at 1 with $1 < \varphi'(0) \leqslant \infty$ and $\varphi'(1) < 1$. Then the constant function 1 is cyclic for V_ψ, where $\psi(x) = 1-\varphi(1-x)$, if and only if the span of the eigenfunctions of V_φ is dense in $L^2[0,1]$.*

In [**2**, **15**], the eigenfunctions and the eigenvalues of several parametric families of Volterra composition operators are calculated. For instance, the spectrum $\sigma(V_{\varphi_\alpha}) = \{(1-\alpha)\alpha^n\}_{n\geqslant 0} \cup \{0\}$ for $\varphi_\alpha(x) = x^\alpha$ with $0 < \alpha < 1$. In this case, all the eigenvalues are simple and the set of eigenfunctions spans $L^2[0,1]$.

3. *Asymptotic behaviour of orbits of quasinilpotent V_φ*

If φ is the identity map, then $V_\varphi = V$ for which there has been much interest on the behaviour of the norms $\|V^n\|_p$, $1 \leqslant p \leqslant \infty$ (see the work by Eveson [**4**, **5**]), where the exact asymptotic behaviour of these norms is obtained using testing functions. We shall see that if $\varphi(x) < x$ for $0 < x < 1$, then the norms $\|V_\varphi^n\|_p$ tend to 0 much faster than $\|V^n\|_p$ as $n \to \infty$.

3.1. *The asymptotic behaviour of* $\|V_\varphi^n\|$

We will be mainly concerned with continuous strictly increasing symbols, since it is necessary for cyclicity of Volterra composition operators (see Section 5). However, most of the proofs in this section still work for non-increasing self-maps.

Let φ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) \leqslant x$ for $0 \leqslant x \leqslant 1$ and $\varphi(1) = 1$. Let $\Omega_1(\varphi) = [0,1]$ and, for each $n \geqslant 2$, consider

$$\Omega_n(\varphi) = \{x \in [0,1]^n : x_1 \leqslant \varphi(x_2), x_2 \leqslant \varphi(x_3), \dots, x_{n-1} \leqslant \varphi(x_n)\}. \tag{3.1}$$

Let μ_n be the n-dimensional Lebesgue measure. We have the following lemma.

LEMMA 3.1. *Let φ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) \leqslant x$ for $0 \leqslant x \leqslant 1$ and $\varphi(1) = 1$. Then $\nu_{n+1}(\varphi) \leqslant \|V_\varphi^n\|_p \leqslant \nu_{n-1}(\varphi)$ for each $n \geqslant 2$ and $1 \leqslant p \leqslant \infty$.*

Proof. Let $\mathbf{1}$ denote the function identically 1 on $[0,1]$. It is clear that $\|V_\varphi^n \mathbf{1}\|_\infty = \|V_\varphi^{n-1}\mathbf{1}\|_1 = (V_\varphi^n \mathbf{1})(1) = \nu_n(\varphi)$ for $n \geqslant 1$. Hence, $\|V_\varphi^n\|_p \geqslant \|V_\varphi^n \mathbf{1}\|_p \geqslant \|V_\varphi^n \mathbf{1}\|_1 = \nu_{n+1}(\varphi)$. We also have $\|V_\varphi^n f\|_\infty \leqslant \|V_\varphi^n \mathbf{1}\|_\infty \|f\|_\infty = \nu_n(\varphi)\|f\|_\infty$ for each $f \in L^\infty[0,1]$. Hence $\|V_\varphi^n f\|_p \leqslant \|V_\varphi^{n-1} V_\varphi f\|_\infty \leqslant \nu_{n-1}(\varphi)\|V_\varphi f\|_\infty \leqslant \nu_{n-1}(\varphi)\|f\|_p$ for each $n \geqslant 2$. Thus, $\nu_{n+1}(\varphi) \leqslant \|V_\varphi^n\|_p \leqslant \nu_{n-1}(\varphi)$ for any $n \geqslant 2$. □

Let φ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) \leqslant x$ for $0 \leqslant x \leqslant 1$. For each positive integer n and each $0 < a < 1$, we set

$$\Omega_n^{a,0}(\varphi) = \{x \in \Omega_n(\varphi) : x_n \leqslant a\} \quad \text{and} \quad \Omega_n^{a,1}(\varphi) = \{x \in \Omega_n(\varphi) : x_1 \geqslant a\}. \tag{3.2}$$

LEMMA 3.2. *Let φ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$, and let $\varphi(1) = 1$ and*

$$\delta_0^+ = \varlimsup_{x\to 0} \frac{\varphi(x)}{x}, \quad \delta_0^- = \varliminf_{x\to 0} \frac{\varphi(x)}{x}, \quad \delta_1^+ = \varlimsup_{x\to 1} \frac{1-x}{1-\varphi(x)}, \quad \delta_1^- = \varliminf_{x\to 1} \frac{1-x}{1-\varphi(x)}. \tag{3.3}$$

Then, for each $0 < a < 1$, we have

$$\varlimsup_{n\to\infty} (\mu_n(\Omega_n^{a,0}(\varphi)))^{1/n^2} \leqslant \sqrt{\delta_0^+}, \quad \varliminf_{n\to\infty} (\mu_n(\Omega_n^{a,0}(\varphi)))^{1/n^2} \geqslant \sqrt{\delta_0^-}, \tag{3.4}$$

$$\varlimsup_{n\to\infty} (\mu_n(\Omega_n^{a,1}(\varphi)))^{1/n^2} \leqslant \sqrt{\delta_1^+}, \quad \varliminf_{n\to\infty} (\mu_n(\Omega_n^{a,1}(\varphi)))^{1/n^2} \geqslant \sqrt{\delta_1^-}. \tag{3.5}$$

In particular, if φ is differentiable at 0 and at 1, where $\varphi'(1) = \infty$ is allowed, then

$$\lim_{n\to\infty} (\mu_n(\Omega_n^{a,0}(\varphi)))^{1/n^2} = \sqrt{\varphi'(0)} \quad \textit{and} \quad \lim_{n\to\infty} (\mu_n(\Omega_n^{a,1}(\varphi)))^{1/n^2} = \sqrt{1/\varphi'(1)}. \tag{3.6}$$

Proof. If $\delta_0^+ = 1$, then the first inequality in (3.4) becomes trivial. Indeed, if we denote by u the identity function, then we have

$$\mu_n(\Omega_n^{a,0}(\varphi)) \leqslant \mu_n(\Omega_n(\varphi)) \leqslant \mu_n(\Omega_n(u)) = \frac{1}{(n+1)!}. \tag{3.7}$$

Thus assume $\delta_0^+ < 1$. We take an arbitrary $\delta_0^+ < b < 1$. Clearly, there exist $0 < \delta < 1$ and a strictly increasing continuous self-map ψ of $[0,1]$ such that $\psi(x) < x$ for $0 < x < 1$, $\psi(x) = bx$ for $0 \leqslant x \leqslant \delta$ and $\psi(x) \geqslant \varphi(x)$ for $0 \leqslant x \leqslant 1$. Since $\psi(x) < x$ for $0 < x < 1$, we find that there is a positive integer k such that $\psi_k(a) \leqslant \delta$, where ψ_k is the kth iterate of ψ. It immediately follows that $\mu_n(\Omega_n^{a,0}(\psi)) \leqslant \mu_{n-k}(\Omega_{n-k}^{\delta,0}(\psi))$ for $n > k$. Since $\psi(x) = bx$ for $0 \leqslant x \leqslant \delta$, we have

$$\mu_j(\Omega_j^{\delta,0}(\psi)) = \int_0^\delta dx_j \int_0^{bx_j} dx_{j-1} \cdots \int_0^{bx_3} dx_2 \int_0^{bx_2} dx_1 = \frac{\delta^j b^{j(j-1)/2}}{j!} \tag{3.8}$$

for each $j \geqslant 1$. Since $\mu_n(\Omega_n^{a,0}(\varphi)) \leqslant \mu_n(\Omega_n^{a,0}(\psi))$ for each $n \geqslant 1$, from (3.8) it follows that $\overline{\lim}_{n\to\infty}(\mu_n(\Omega_n^{a,0}(\varphi)))^{1/n^2} \leqslant \sqrt{b}$. Since $\delta_0^+ < b < 1$ was arbitrary, the first inequality in (3.4) follows.

If $\delta_0^- = 0$, then the second inequality in (3.4) is trivial. Thus assume $\delta_0^- > 0$. We take an arbitrary $0 < b < \delta_0^-$. Clearly, there is $0 < \delta < a$ and a strictly increasing continuous self-map ψ of $[0, 1]$ such that $\psi(x) < x$ for $0 < x < 1$, $\psi(x) = bx$ for $0 \leqslant x \leqslant \delta$ and $\psi(x) \leqslant \varphi(x)$ for $0 \leqslant x \leqslant 1$. Since $\mu_n(\Omega_n^{a,0}(\varphi)) \geqslant \mu_n(\Omega_n^{a,0}(\psi)) \geqslant \mu_n(\Omega_n^{\delta,0}(\psi))$ for each $n \geqslant 1$, from (3.8) we obtain

$$\varliminf_{n\to\infty} (\mu_n(\Omega_n^{a,0}(\varphi)))^{1/n^2} \geqslant \sqrt{b}.$$

Since $0 < b < \delta_0^-$ was arbitrary, the second inequality in (3.4) also follows.

Finally, since φ satisfies (3.5) if and only if $\phi(x) = 1 - \varphi^{-1}(1 - x)$ satisfies (3.4), the proof of the statement of the lemma is complete. □

In order to state the main result of this section, we consider

$$\phi(u, v) = \begin{cases} \exp\left(\dfrac{\ln u \ln v}{2\ln(uv)}\right) & \text{if } u > 0, v > 0 \text{ and } (u, v) \neq (1, 1), \\ \sqrt{|u - v|} & \text{if } u = 0 \text{ or } v = 0, \\ 1 & \text{if } (u, v) = (1, 1), \end{cases}$$

which is clearly continuous on $[0, 1]^2$ and takes its values in $[0, 1]$.

THEOREM 3.3. *Let φ be a continuous strictly increasing self-map of $[0, 1]$ with $\varphi(x) < x$ for $0 < x < 1$, and let $\varphi(1) = 1$ and $\delta_0^+, \delta_0^-, \delta_1^+, \delta_1^-$ be as in (3.3). Then, for $1 \leqslant p \leqslant \infty$, we have*

$$\rho_- \leqslant \varliminf_{n\to\infty} \|V_\varphi^n\|_p^{1/n^2} \leqslant \varlimsup_{n\to\infty} \|V_\varphi^n\|_p^{1/n^2} \leqslant \rho_+,$$

where $\rho_- = \phi(\delta_0^-, \delta_1^-)$ and $\rho_+ = \phi(\delta_0^+, \delta_1^+)$. In particular, if φ is differentiable at 0 and at 1, then

$$\lim_{n\to\infty} \|V_\varphi^n\|^{1/n^2} = \phi(\varphi'(0), 1/\varphi'(1)).$$

Proof. According to Lemma 3.1, it is enough to show that

$$\rho_- \leqslant \varliminf_{n\to\infty} (\nu_n(\varphi))^{1/n^2} \leqslant \varlimsup_{n\to\infty} (\nu_n(\varphi))^{1/n^2} \leqslant \rho_+. \tag{3.9}$$

If $\rho_+ = 1$, then the last inequality in (3.9) follows from the second one in (3.7). Thus assume that $\rho_+ < 1$. Hence, we must have $\delta_0^+ < 1$ and $\delta_1^+ < 1$. We take $\delta_0^+ < b_0 < 1$ and $\delta_1^+ < b_1 < 1$. By Lemma 3.2, there is $c > 0$ such that

$$\mu_n(\Omega_n^{1/2,0}(\varphi)) \leqslant c b_0^{n^2/2} \quad \text{and} \quad \mu_n(\Omega_n^{1/2,1}(\varphi)) \leqslant c b_1^{n^2/2} \tag{3.10}$$

for each positive integer n. Clearly, $\Omega_n(\varphi) \subset \bigcup_{k=0}^n A_k$, where $A_0 = \Omega_n^{1/2,0}(\varphi)$, $A_n = \Omega_n^{1/2,1}(\varphi)$ and $A_k = \Omega_{n-k}^{1/2,0}(\varphi) \times \Omega_k^{1/2,1}(\varphi)$ for $0 < k < n$. Hence,

$$\nu_n(\varphi) \leqslant \sum_{k=0}^n \mu_n(A_k) = \sum_{k=0}^n \mu_{n-k}(\Omega_{n-k}^{1/2,0}(\varphi))\mu_k(\Omega_k^{1/2,1}(\varphi)).$$

Using (3.10), we obtain

$$\begin{aligned}\nu_n(\varphi) &\leqslant c^2 \sum_{k=0}^{n} b_0^{(n-k)^2/2} b_1^{k^2/2} \\ &\leqslant c^2(n+1) \max_{0\leqslant k\leqslant n} b_0^{(n-k)^2/2} b_1^{k^2/2} \\ &\leqslant c^2(n+1) \left(\max_{[0,1]} b_0^{(1-x)^2/2} b_1^{x^2/2} \right)^{n^2}.\end{aligned}$$

The last maximum is attained for $x = \ln b_0 / \ln(b_0 b_1)$ and equals $\phi(b_0, b_1)$. Therefore, $\overline{\lim}_{n\to\infty} (\nu_n(\varphi))^{1/n^2} \leqslant \phi(b_1, b_2)$. Since $\delta_0^+ < b_0 < 1$ and $\delta_1^+ < b_1 < 1$ were arbitrary, the last inequality of (3.9) is satisfied.

If $\rho_- = 0$, then the first inequality in (3.9) is trivial. Thus assume $\rho_- > 0$. Hence, we must have $\delta_0^- > 0$ and $\delta_1^- > 0$. We take $0 < b_0 < \delta_0^-$ and $0 < b_1 < \delta_1^-$. Let $a > 0$ be small enough to ensure that $a < \varphi(1-a)$. By Lemma 3.2, there is $c > 0$ such that

$$\mu_n(\Omega_n^{a,0}(\varphi)) \geqslant c b_0^{n^2/2} \quad \text{and} \quad \mu_n(\Omega_n^{1-a,1}(\varphi)) \geqslant c b_1^{n^2/2} \tag{3.11}$$

for each $n \geqslant 1$. Choose a sequence $\{k_n\}_{n\geqslant 1}$ of positive integers such that $k_n < n$ for each n and k_n/n tends to $\ln b_1/\ln(b_1 b_2)$ as n tends to ∞. Clearly, $\Omega_n(\varphi) \supset A = \Omega_{n-k_n}^{a,0}(\varphi) \times \Omega_{k_n}^{1-a,1}(\varphi)$. Hence, $\nu_n(\varphi) \geqslant \mu_n(A) = \mu_{n-k_n}(\Omega_{n-k_n}^{a,0}(\varphi))\mu_{k_n}(\Omega_{k_n}^{1-a,1}(\varphi))$. Using (3.11), we obtain

$$\nu_n(\varphi) \geqslant c^2 b_0^{(n-k_n)^2/2} b_1^{k_n^2/2} = c^2 (b_0^{(1-(k_n/n))^2} b_1^{(k_n/n)^2})^{n^2/2}.$$

Since k_n/n tends to $\ln b_0/\ln(b_0 b_1)$, we see that $\lim_{n\to\infty} b_0^{1-k_n/n} b_1^{k_n/n} = \phi(b_0, b_1)$. Thus we obtain $\underline{\lim}_{n\to\infty} (\nu_n(\varphi))^{1/n^2} \geqslant \phi(b_0, b_1)$. Since $0 < b_0 < \delta_0^-$ and $0 < b_1 < \delta_1^-$ were arbitrary, the first inequality in (3.9) also holds. The proof is complete. □

COROLLARY 3.4. *Let φ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$, and let $\varphi(1) = 1$ and φ be differentiable at 0 and 1. If $\varphi'(0) = 0$, then $\lim_{n\to\infty} \|V_\varphi^n\|_p^{1/n^2} = 1/\sqrt{\varphi'(1)}$ and if $\varphi'(1) = \infty$, then $\lim_{n\to\infty} \|V_\varphi^n\|_p^{1/n^2} = \sqrt{\varphi'(0)}$ for $1 \leqslant p \leqslant \infty$.*

3.2. *Orbits of V_φ: upper estimate*

The next lemma will be very useful to determine the cyclic properties of V_φ.

LEMMA 3.5. *Let φ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$, and let $\varphi(1) = 1$ and*

$$\delta_1^+ = \delta_1^+(\varphi) = \overline{\lim_{x\to 1}} \frac{1-x}{1-\varphi(x)}.$$

Assume also that f in $L^p[0,1]$, $1 \leqslant p \leqslant \infty$, satisfies $\inf \operatorname{supp}(f) > 0$. *Then, $\overline{\lim}_{n\to\infty} \|V_\varphi^n f\|_p^{1/n^2} \leqslant \sqrt{\delta_1^+}$. In particular, for φ differentiable at 1, we have that $\overline{\lim}_{n\to\infty} \|V_\varphi^n f\|_p^{1/n^2} \leqslant \sqrt{1/\varphi'(1)}$.*

Proof. Let $\varepsilon > 0$ be such that f vanishes on $[0,\varepsilon]$. Since $V_\varphi f$ is continuous and also vanishes $[0,\varepsilon]$, there is $c > 0$ for which $|(V_\varphi f)(x)| \leqslant c\chi_{[\varepsilon,1]}(x)$ for each $0 \leqslant x \leqslant 1$, where $\chi_{[\varepsilon,1]}$ is the characteristic function of $[\varepsilon, 1]$. Hence,

$$\|V_\varphi^n f\|_p \leqslant \|V_\varphi^n f\|_\infty \leqslant c\|V_\varphi^n \chi_{[\varepsilon,1]}\|_\infty = c(V_\varphi^n \chi_{[\varepsilon,1]})(1). \tag{3.12}$$

Let $\Omega_n^{\varepsilon,1}(\varphi)$ be as in (3.2). Then $(V_\varphi^n \chi_{[\varepsilon,1]})(1) = \mu_n(\Omega_n^{\varepsilon,1}(\varphi))$ for each positive integer n. Therefore, by Lemma 3.2, we have

$$\varlimsup_{n\to\infty} \|V_\varphi^n \chi_{[\varepsilon,1]}\|_1^{1/n^2} = \varlimsup_{n\to\infty} (\mu_n(\Omega_n^{\varepsilon,1}(\varphi)))^{1/n^2} \leqslant \sqrt{\delta_1^+}.$$

The required result follows immediately from the previous inequalities. □

3.3. *The backward orbits of V_φ*

In this subsection, we consider the asymptotic behaviour of certain backward orbits of V_φ. In the next two sections, we shall apply these results to determine the cyclic behaviour of Volterra composition operators.

If S is any linear operator acting on a linear space X, then $S^\infty(X) = \bigcap_{n=0}^\infty S^n(X)$ is a subspace of X invariant under S. Moreover, since $S(S^\infty(X)) = S^\infty(X)$, the restriction of S to $S^\infty(X)$ is always onto. In addition, if $\ker S = \{0\}$, then S is one-to-one from $S^\infty(X)$ onto itself. Thus the backward orbits of any x in $S^\infty(X)$ are well defined. This is in particular our case when $\ker V_\varphi = \{0\}$.

Recall that $\mathcal{C}_0[0,1]$ is the subspace of $\mathcal{C}[0,1]$ of functions vanishing at 0, endowed with the supremum norm.

Theorem 3.6. *Let φ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$ and $\varphi(1) = 1$. Assume also that φ is analytic at 0 and $\varphi'(0) > 0$. Then, for each $b > 1/\varphi'(0)$, the set $F_b = \{f \in V_\varphi^\infty(\mathcal{C}_0[0,1])$ such that $\varlimsup_{n\to\infty}\|V_\varphi^{-n} f\|_\infty^{1/n^2} \leqslant \sqrt{b}\}$ is a dense linear manifold of $\mathcal{C}_0[0,1]$ satisfying $V_\varphi(F_b) = V_\varphi^{-1}(F_b) = F_b$.*

The remainder of this section is devoted to showing Theorem 3.6. In order to do so, we need the following well-known criterion of analyticity. For each $f \in \mathcal{C}^\infty[a,b]$ we set

$$M_n(f) = \frac{1}{n!}\max_{[a,b]} |f^{(n)}|.$$

Then f is analytic on $[u,v]$ if and only if

$$\varlimsup_{n\to\infty} (M_n(f))^{1/n} < \infty. \tag{3.13}$$

We also need the space $\mathcal{F}[a,b]$ defined as

$$\left\{ f \in \mathcal{C}^\infty[a,b] : f^{(n)}(a) = f^{(n)}(b) = 0 \ \text{ for each } n \geqslant 0 \text{ and } \varlimsup_{n\to\infty} (M_n(f))^{1/n^2} \leqslant 1 \right\}.$$

By means of Leibniz's formula for the derivatives of a product, it is easy to check that if f is in $\mathcal{F}[a,b]$ and g in $\mathcal{C}^\infty[a,b]$ satisfies $\varlimsup_{n\to\infty}(M_n(g))^{1/n^2} \leqslant 1$, then fg belongs to $\mathcal{F}[a,b]$. In particular, the space $\mathcal{F}[a,b]$ is an algebra with respect to pointwise multiplication and is invariant under multiplication by analytic functions.

Lemma 3.7. *Assume that $-\infty < a < b < \infty$. Then*

$$h(x) = \begin{cases} e^{-1/(x-a)-1/(b-x)} & \text{if } a < x < b, \\ 0 & \text{if } x = a \text{ or } x = b \end{cases} \tag{3.14}$$

belongs to $\mathcal{F}[a,b]$.

Proof. Clearly, h is in $\mathcal{C}^\infty[a,b]$ with $h^{(n)}(a)=h^{(n)}(b)=0$ for each $n\geqslant 0$. Thus we need only prove that

$$\varlimsup_{n\to\infty}(M_n(h))^{1/n^2}\leqslant 1. \tag{3.15}$$

To this end, we estimate $M_n(g)$, where $g(x)=e^{-1/x}$ for $x>0$ and $g(0)=0$. By induction, one easily sees that $g^{(n)}(x)=p_n(1/x)g(x)$ for $x\neq 0$, where

$$p_0=1 \quad\text{and}\quad p_{n+1}(t)=t^2p_n(t)-t^2p_n'(t)\ \text{ for each } n\geqslant 1. \tag{3.16}$$

Clearly, $p_n(t)=\sum_{j=0}^{2n}a_{n,j}t^j$, where the coefficients $a_{n,j}$ are real. Therefore,

$$M_n(g)=\frac{1}{n!}\sup_{x>0}|g^{(n)}(x)|\leqslant\frac{1}{n!}\sum_{j=0}^{2n}|a_{n,j}|\sup_{x>0}x^{-j}e^{-1/x}=\frac{1}{n!}\sum_{j=0}^{2n}|a_{n,j}|\sup_{x>0}x^je^{-x}.$$

Since $\sup_{x>0}x^je^{-x}=(j/e)^j$, we have $n!M_n(g)\leqslant\sum_{j=0}^{2n}|a_{n,j}|(j/e)^j\leqslant(2n/e)^{2n}\sigma_n$, where $\sigma_n=\sum_{j=0}^{2n}|a_{n,j}|$. Using (3.16), we see that $\sigma_{n+1}\leqslant(2n+1)\sigma_n$ and, therefore, $\sigma_n\leqslant(2n)!/(2^nn!)$. Upon putting everything together and using Stirling's formula, one easily sees that $M_n(g)\leqslant(2n/e)^{2n}$.

Since $h(x)=g(x-a)g(b-x)$, applying Leibniz's formula, we see that

$$h^{(n)}(x)=\sum_{k=0}^{n}\frac{(-1)^kn!}{k!(n-k)!}g^{(k)}(b-x)g^{(n-k)}(x-a)\quad\text{for } a\leqslant x\leqslant b.$$

Therefore, using that $M_k(g)\leqslant(2k/e)^{2k}$ in the second inequality below, we have

$$M_n(h)\leqslant\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}M_k(g)M_{n-k}(g)\leqslant 2^n\max_{0\leqslant k\leqslant n}(2k)^{2k}(2n-2k)^{2n-2k}<(2n)^{2n},$$

from which (3.15) follows and the result is proved. □

The following lemma can also be derived from the Denjoy–Carleman theorem (see [**18**, p. 380]). Here, we provide an easy elementary proof. We denote by $\mathcal{C}_{00}[a,b]$ the Banach subspace of $\mathcal{C}[a,b]$ that consists of functions that vanish at a and b.

LEMMA 3.8. *Assume that $a<b$ are real. Then $\mathcal{F}[a,b]$ is dense in $\mathcal{C}_{00}[a,b]$ and $\mathcal{F}^+[a,b]=\{f\in\mathcal{F}[a,b]$ such that $f(x)\geqslant 0$ for each $x\in[a,b]\}$ is dense in $\mathcal{C}_{00}^+[a,b]=\{f\in\mathcal{C}_{00}[a,b]$ such that $f(x)\geqslant 0$ for each $x\in[a,b]\}$.*

Proof. Let h be the function in (3.14). By Lemma 3.7, we know that h is in $\mathcal{F}[a,b]$. Since h is in $\mathcal{C}_{00}[a,b]$ and $h(x)>0$ for $a<x<b$, we see that the set W consisting of functions ph such that p is a polynomial is dense in $\mathcal{C}_{00}[a,b]$. Also, the set W^+ consisting of functions $g\in W$ such that $g(x)\geqslant 0$ on $[a,b]$ is dense in $\mathcal{C}_{00}^+[a,b]$. Now, the result follows because $\mathcal{F}[a,b]$ is stable with respect to multiplication by polynomials and therefore $W\subset\mathcal{F}[a,b]$ and $W^+\subset\mathcal{F}^+[a,b]$. □

We need an operator related to the local inverse of V_φ. For $0<a<1$, we set

$$\mathcal{E}_a=\{f\in\mathcal{C}^\infty[0,1]\text{ such that supp } f\subset[0,a]\text{ and } f^{(n)}(0)=0\text{ for } n\geqslant 0\}. \tag{3.17}$$

For an analytic function $\psi:[0,a]\to[0,1]$ such that $\psi(a)\geqslant a$ and $\psi(0)=0$, consider the operator $T_\psi:\mathcal{E}_a\to\mathcal{E}_a$ defined as

$$(T_\psi f)(x)=\begin{cases} f'(\psi(x)) & \text{if } x\in[0,a],\\ 0 & \text{if } x\in(a,1].\end{cases} \tag{3.18}$$

The requirements $\psi(a) \geqslant a$ and $\psi(0) = 0$ imply that T_ψ acts from $\mathcal{E}_a$ into itself. As usual, for each pair n and l of non-negative integers, we write $(n)_l = 1$ for $l = 0$ and $(n)_l = (n+1)\dots(n+l)$ for $l > 0$.

To prove Lemmas 3.10, 3.13 and 3.15, we need the Faá di Bruno formula for the nth derivative of a type (see [**15**, Lemma 6.1] or [**21**, Chapter 3]).

Lemma 3.9. *Let f and g be in $\mathcal{C}^n[u,v]$. Then, for each $u \leqslant x \leqslant v$, we have*

$$(g \circ f)^{(n)}(x) = n! \sum_{k_1+\dots+nk_n=n} \frac{g^{(k_1+\dots+k_n)}(f(x))}{k_1!\dots k_n!(1!)^{k_1}\dots(n!)^{k_n}} (f'(x))^{k_1}\dots(f^{(n)}(x))^{k_n}. \tag{3.19}$$

Here $k_1+\dots+nk_n$ indicates that the sum runs through all the n-tuples such that $\sum_{j=1}^n jk_j = n$. An immediate consequence of the above lemma (see, for instance, [**15**, Section 6]) is that, for each $c \in \mathbb{C}$ and each $n \geqslant 1$, the following holds:

$$\sum_{k_1+\dots+nk_n=n} \frac{(k_1+\dots+k_n)!}{k_1!\dots k_n!} c^{k_1+\dots+k_n} = c(c+1)^{n-1}. \tag{3.20}$$

Lemma 3.10. *Let ψ be an analytic function from $[0,a]$ into $[0,1]$, where $0 < a < 1$, with $\psi(0) = 0$ and $\psi(a) \geqslant a$. Let $\{c_n\}_{n\geqslant 0}$ be such that $c_n \geqslant 1$ with $\lim_{n\to\infty} c_n^{1/n} = 1$ and let $\{f_n\}_{n\geqslant 0}$ be in $\mathcal{E}_a$ satisfying $\beta_n = \sup_{k\geqslant 0} M_n(f_k)c_k^{-n-1} < \infty$, for each $n \geqslant 0$ and $\overline{\lim}_{n\to\infty}\beta_n^{1/n^2} \leqslant 1$. Then $\overline{\lim}_{n\to\infty}\|T_\psi^n f_n\|_\infty^{1/n^2} \leqslant \sqrt{\gamma}$.*

Proof. The proof is split into three steps.

Step 1. *Let $\{\widehat{\beta}_n\}_{n\geqslant 0}$ be a sequence such that $\{\widehat{\beta}_n^{1/n}\}$ is increasing. Assume also that $c > 0$ and l is a non-negative integer. Then, for f in $\mathcal{E}_a$ satisfying $M_n(f) \leqslant c\,(n)_l\widehat{\beta}_n$ for each $n \geqslant 0$, we have*

$$M_n(T_\psi f) \leqslant c\,(n)_{l+1}\gamma^n\widehat{\beta}_{n+1}\left(1 + \frac{R}{\gamma\widehat{\beta}_{n+1}^{1/(n+1)}}\right)^n \quad \textit{for each } n \geqslant 0,$$

where

$$R = \sup_{n\geqslant 2}\left(\frac{M_n(\psi)}{\gamma}\right)^{1/(n-1)}. \tag{3.21}$$

Proof of Step 1. Since ψ is analytic on $[0,a]$, by (3.13), we see that R is finite. Clearly, $\|T_\psi f\|_\infty \leqslant \|f'\|_\infty = M_1(f) \leqslant c(1)_l\widehat{\beta}_1 = c(0)_{l+1}\widehat{\beta}_1$. Thus the result is true for $n = 0$. Since

$$M_n(\psi) \leqslant \gamma R^{n-1} \quad \text{for} \quad n \geqslant 1, \tag{3.22}$$

using Lemma 3.9, for each $0 \leqslant x \leqslant a$ with $0 < \psi(x) < a$ and for each $n \geqslant 1$, we have

$$(T_\psi f)^{(n)}(x) = (f' \circ \psi)^{(n)}(x) = n! \sum_{n=k_1+\dots+nk_n} \frac{f^{(k_1+\dots+k_n+1)}(\psi(x))}{k_1!\dots k_n!(1!)^{k_1}\dots(n!)^{k_n}} (\psi'(x))^{k_1}\dots(\psi^{(n)}(x))^{k_n}.$$

From (3.18), we have $(T_\psi f)^{(n)}(x) = 0$ for f in $\mathcal{E}_a$ and $\psi(x) \geqslant a$ and, therefore, we may write

$$\begin{aligned} M_n(T_\psi f) &\leqslant \sum_{n=k_1+\dots+nk_n} \frac{(k_1+\dots+k_n+1)!}{k_1!\dots k_n!} M_{k_1+\dots+k_n+1}(f)(M_1(\psi))^{k_1}\dots(M_n(\psi))^{k_n} \\ &\leqslant (n+1) \sum_{n=k_1+\dots+nk_n} \frac{(k_1+\dots+k_n)!}{k_1!\dots k_n!} M_{k_1+\dots+k_n+1}(f)(M_1(\psi))^{k_1}\dots(M_n(\psi))^{k_n}. \end{aligned}$$

From (3.22) and the fact that $M_k(f) \leqslant c(k)_l \widehat{\beta}_k$, we have

$$M_n(T_\psi f) \leqslant c(n)_{l+1} \sum_{n=k_1+\ldots+nk_n} \frac{(k_1+\ldots+k_n)!}{k_1!\ldots k_n!} \widehat{\beta}_{k_1+\ldots+k_n+1} \gamma^{k_1} (\gamma R)^{k_2} \ldots (\gamma R^{n-1})^{k_n}.$$

Since $\{\widehat{\beta}_k^{1/k}\}$ is increasing, it follows that $\widehat{\beta}_k \leqslant (\widehat{\beta}_m)^{k/m}$ for $1 \leqslant k \leqslant m$. Therefore,

$$M_n(T_\psi f) \leqslant c\,(n)_{l+1} R^n \widehat{\beta}_{n+1}^{1/(n+1)} \sum_{n=k_1+\ldots+nk_n} \frac{(k_1+\ldots+k_n)!}{k_1!\ldots k_n!} \left(\frac{\gamma \widehat{\beta}_{n+1}^{1/(n+1)}}{R} \right)^{k_1+\ldots+k_n}.$$

Applying (3.20), we have

$$\begin{aligned} M_n(T_\psi f) &\leqslant c(n)_{l+1} \gamma R^{n-1} \widehat{\beta}_{n+1}^{2/(n+1)} \left(1 + \frac{\gamma \widehat{\beta}_{n+1}^{1/(n+1)}}{R} \right)^{n-1} \\ &= c(n)_{l+1} \gamma^n \widehat{\beta}_{n+1} \left(1 + \frac{R}{\gamma \widehat{\beta}_n^{1/(n+1)}} \right)^{n-1} \\ &\leqslant c(n)_{l+1} \gamma^n \widehat{\beta}_{n+1} \left(1 + \frac{R}{\gamma \widehat{\beta}_n^{1/(n+1)}} \right)^{n}. \end{aligned}$$

The proof of Step 1 is complete.

Step 2. Under the hypotheses of the lemma, we have that $\widetilde{\beta}_n = \sup_{k\geqslant 0} M_n(T_\psi f_k) c_k^{-n-2}$ *is finite for each* $n \geqslant 0$ *and* $\overline{\lim}_{n\to\infty} \widetilde{\beta}_n^{1/n^2} \leqslant 1$.

Proof of Step 2. Let $\delta > 1$ be fixed. Since $\overline{\lim}_{n\to\infty} \beta_n^{1/n^2} \leqslant 1$, we may choose $C > 0$ such that $\beta_n \leqslant C\,\delta^{n^2}$ for each $n \geqslant 0$. For each $n \geqslant 0$, we set $\widehat{\beta}_n = c_k^n \delta^{n^2}$, where $c = Cc_k$. Then, for each $k \geqslant 0$, we have $M_n(f_k) \leqslant c\widehat{\beta}_n$. By Step 1, we have

$$M_n(T_\psi f_k) \leqslant c(n+1) \gamma^n \widehat{\beta}_{n+1} \left(1 + \frac{R}{\gamma \widehat{\beta}_{n+1}^{1/(n+1)}} \right)^n \quad \text{for each } n \geqslant 0.$$

Upon substituting the values of c and $\widehat{\beta}_{n+1}$, we obtain

$$\begin{aligned} M_n(T_\psi f_k) &\leqslant C\,(n+1) c_k^{n+2} \gamma^n \delta^{(n+1)^2} \left(1 + \frac{R}{\gamma c_k \delta^{n+1}} \right)^n \\ &\leqslant C\,(n+1) c_k^{n+2} \gamma^n \delta^{(n+1)^2} \left(1 + \frac{R}{\gamma \delta^{n+1}} \right)^n. \end{aligned}$$

Therefore,

$$\widetilde{\beta}_n \leqslant C\,(n+1) \gamma^n \delta^{(n+1)^2} \left(1 + \frac{R}{\gamma \delta^{n+1}} \right)^n.$$

Thus $\overline{\lim}_{n\to\infty} \widetilde{\beta}_n^{1/n^2} \leqslant \delta$. Since $\delta > 1$ was arbitrary, the proof of Step 2 is complete.

Step 3. The conclusion of the lemma holds.

Proof of Step 3. Let $\delta > 1$ be fixed. Since $\psi(0) = 0$ and $\psi(a) \geqslant a$, we see that $\gamma = \|\psi'\|_\infty \geqslant 1$. Thus using that $c_j \geqslant 1$, we may take a positive integer l such that

$$\delta^{n/2} \left(1 + \frac{R}{\gamma (\gamma\delta)^m \delta^{(n+1)/4} c_j} \right)^n \leqslant \delta^n \quad \text{for each } m \geqslant l \text{ and } n, j \geqslant 0. \tag{3.23}$$

Indeed, it is enough to take l with $\delta^l \geqslant R/(\sqrt{\delta}-1)$. As in the proof of Step 2, there is $C>0$ such that, for $0 \leqslant k \leqslant l$, we have

$$M_n(T_\psi^k f_j) \leqslant C c_j^{n+k+1} \delta^{k/4} (n)_k (\gamma\delta)^{k(n+(k-1)/2)} \delta^{n^2/4} \quad \text{for each } n, j \geqslant 0. \tag{3.24}$$

We will prove that (3.24) also holds for each $k \geqslant l+1$. Suppose that (3.24) is true for an integer $k=m \geqslant l$. For $k=m$, we can rewrite (3.24) as $M_n(T_\psi^m f_j) \leqslant c(n)_m \widehat{\beta}_n$, where $c = C\, c_j^{m+1} \delta^{m/4} (\gamma\delta)^{m(m-1)/2}$ and $\widehat{\beta}_n = c_j^n (\gamma\delta)^{mn} \delta^{n^2/4}$. Applying Step 1, we have

$$M_n(T_\psi^{m+1} f_j) \leqslant c\,(n)_{m+1} \gamma^n \widehat{\beta}_{n+1} \left(1 + \frac{R}{\gamma \widehat{\beta}_{n+1}^{1/(n+1)}}\right)^n,$$

which is equal to

$$C c_j^{m+n+1} \delta^{(m+1)/4} (n)_{m+1} (\gamma\delta)^{m(m-1)/2} \gamma^n (\gamma\delta)^{mn+m} \delta^{n^2/4} \delta^{n/2} \left(1 + \frac{R}{\gamma c_j (\gamma\delta)^m \delta^{(n+1)/4}}\right)^n.$$

Since $m \geqslant l$, we may use (3.23) to obtain

$$\begin{aligned} M_n(T_\psi^{m+1} f) &\leqslant C c_j^{m+n+1} \delta^{(m+1)/4} (n)_{m+1} (\gamma\delta)^{m(m-1)/2} \gamma^n (\gamma\delta)^{mn+m} \delta^{n^2/4} \delta^n \\ &= C c_j^{n+m+1} \delta^{(m+1)/4} (n)_{m+1} (\gamma\delta)^{mn+m(m-1)/2+n+m} \delta^{n^2/4} \\ &= C \delta^{(m+1)/4} (n)_{m+1} (\gamma\delta)^{(m+1)(n+(m/2))} \delta^{n^2/4}, \end{aligned}$$

which is (3.24) for $k=m+1$. Thus (3.24) holds for all non-negative integers k, n and j. For $n=0$ and $j=k$, we find that (3.24) implies that $\|T_\psi^k f_k\|_\infty \leqslant C c_k^{k+1} \delta^{k/4} k! (\gamma\delta)^{k(k-1)/2}$. Since $c_k^{1/k}$ tends to 1, we obtain $\overline{\lim}_{k\to\infty} \|T_\psi^k f_k\|_\infty^{1/k^2} \leqslant \sqrt{\gamma\delta}$. Since $\delta > 1$ was arbitrary, it follows that $\overline{\lim}_{k\to\infty} \|T_\psi^k f_k\|_\infty^{1/k^2} \leqslant \sqrt{\gamma}$, which is the required result. The proof of Step 3 and that of the statement of the lemma are complete. □

Observe that the formula for the adjoint of V_φ is

$$(V_\varphi^\star f)(x) = \int_{\varphi_{-1}(x)}^1 f(t)\, dt.$$

which, as an operator, makes sense on $L^p[0,1]$ for $1 \leqslant p \leqslant \infty$. Indeed, the adjoint of $V_\varphi^\star$ acting on $L^1[0,1]$ is V_φ acting on $L^\infty[0,1]$. The next lemma, which will be very useful, describes the behaviour of the supports of the iterates $\{V_\varphi^n f\}$ and $\{V_\varphi^{\star n} f\}$. The proof, which is straightforward, is omitted.

Lemma 3.11. *Let φ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ and $\varphi(1) = 1$. Assume also that f is in $L^1[0,1]$. Then we have*

(a) $\inf \operatorname{supp} (V_\varphi f) = \varphi_{-1}(\inf \operatorname{supp} (f))$;
(b) $\sup \operatorname{supp} (V_\varphi f) \in \{1, \varphi_{-1}(\sup \operatorname{supp} (f))\}$;
(c) $\inf \operatorname{supp} (V_\varphi^\star f) \in \{0, \varphi(\inf \operatorname{supp} (g))\}$;
(d) $\sup \operatorname{supp} (V_\varphi^\star g) = \varphi(\sup \operatorname{supp} (g))$;
(e) $\sup \operatorname{supp} (V_\varphi^n f)$ *tends to* 1 *and* $\inf \operatorname{supp} (V_\varphi^{\star n} g)$ *tends to* 0 *as* n *tends to* ∞.

When dealing with supercyclicity of V_φ in Section 5, we shall need special dense subsets of $\mathcal{C}_0[0,1]$. For each $0 < a < 1$, we set $\mathcal{C}_a = \{f \in \mathcal{C}_0[0,1] \text{ such that } \sup \operatorname{supp} (f) \leqslant a\}$. We have the following lemma.

LEMMA 3.12. *Let φ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$ and $\varphi(1) = 1$. Assume that $0 < a < 1$. Then $Z = \operatorname{span}\left(\bigcup_{n=0}^{\infty} V_\varphi^n(\mathcal{C}_a)\right)$ is dense in $\mathcal{C}_0[0,1]$.*

Proof. It is enough to prove that Z is dense in $L^2[0,1]$. Indeed, once this is proved, the result follows because V_φ acting from $L^2[0,1]$ into $\mathcal{C}_0[0,1]$ is bounded with dense range and the image of a dense set under an operator with dense range is itself dense and Z is invariant under V_φ.

Thus assume that Z is not dense in $L^2[0,1]$. Then there is a non-zero g in $L^2[0,1]$ such that $\langle V_\varphi^n f, g\rangle = \langle f, V_\varphi^{\star n} g\rangle = 0$ for each f in $\mathcal{C}_a$ and for each $n \geqslant 0$. This means that $\inf\operatorname{supp}(V_\varphi^{\star n} g) \geqslant a$ for each $n \geqslant 0$. Now, by Lemma 3.11, we have $\inf\operatorname{supp}(V_\varphi^{\star n} g)$ tends to 0 as n tends to ∞, which is a contradiction. □

For $0 < a < 1$, we shall write

$$\mathcal{F}_a = \left\{ f \in \mathcal{E}_a \text{ such that } \overline{\lim_{n\to\infty}}\,(M_n(f))^{1/n^2} \leqslant 1 \right\}, \tag{3.25}$$

where $\mathcal{E}_a$ is as defined in (3.17); that is, f belongs to $\mathcal{F}_a$ if and only if f belongs to $\mathcal{C}^\infty[0,1]$, $\operatorname{supp}(f) \subseteq [0,a]$ and the restriction of f to $[0,a]$ belongs to $\mathcal{F}[0,a]$.

LEMMA 3.13. *Let ψ be analytic from $[0,a]$ into $[0,1]$, where $0 < a < 1$, with $\psi(0) = 0$ and $\psi(a) \geqslant a$. Let T_ψ be the operator on $\mathcal{E}_a$ defined in (3.18) and C_ψ be the operator on $\mathcal{E}_a$ defined as*

$$(C_\psi f)(x) = \begin{cases} f(\psi(x)) & \text{if } x \in [0,a], \\ 0 & \text{if } x \in (a,1]. \end{cases}$$

Then $\mathcal{F}_a$ is invariant both under C_ψ and T_ψ.

Proof. Let $\gamma > 1$ be fixed. If f is in $\mathcal{F}_a$, then there is $c \geqslant 1$ such that $M_n(f) \leqslant c^n \gamma^{n^2}$ for each positive integer n. Since ψ is analytic, by (3.13), the value $R = \sup_{n\geqslant 2}(M_n(\psi)/\gamma)^{1/(n-1)}$ is finite. Now, from Lemma 3.9, it follows that

$$M_n(C_\psi f) \leqslant \sum_{n=k_1+\ldots+nk_n} \frac{(k_1+\ldots+k_n)!}{k_1!\ldots k_n!} M_{k_1+\ldots+k_n}(f)(M_1(\psi))^{k_1}\ldots(M_n(\psi))^{k_n}.$$

Using that $M_m(\psi) \leqslant \gamma R^{m-1}$ and $M_k(f) \leqslant c^k\gamma^{k^2}$, we obtain

$$\begin{aligned} M_n(C_\psi f) &\leqslant \sum_{n=k_1+\ldots+nk_n} \frac{(k_1+\ldots+k_n)!}{k_1!\ldots k_n!} c^{k_1+\ldots+k_n}\gamma^{(k_1+\ldots+k_n)^2}\gamma^{k_1}(\gamma R)^{k_2}\ldots(\gamma R^{n-1})^{k_n} \\ &\leqslant R^n \sum_{n=k_1+\ldots+nk_n} \frac{(k_1+\ldots+k_n)!}{k_1!\ldots k_n!}\left(\frac{c\gamma^{n+1}}{R}\right)^{k_1+\ldots+k_n}. \end{aligned}$$

Upon applying (3.20), we obtain $M_n(C_\psi f) \leqslant R^{n-1}c\gamma^{n+1}(1 + c\gamma^{n+1}/R)^{n-1} \leqslant (R + c\gamma^{n+1})^n$. Therefore, it follows that $\overline{\lim}_{n\to\infty}(M_n(C_\psi f))^{1/n^2} \leqslant \gamma$. Since $\gamma > 1$ was arbitrary, we see that $\overline{\lim}_{n\to\infty}(M_n(C_\psi f))^{1/n^2} \leqslant 1$ and, therefore, $C_\psi f$ belongs to $\mathcal{F}_a$. Finally, it is clear that $\mathcal{F}_a$ is also invariant under the differentiation operator $Df = f'$. Since $T_\psi = C_\psi D$, the result follows. □

The next lemma is needed not only to prove Theorem 3.6, but also to show the non-cyclicity of certain Volterra composition operators in Section 5.

Lemma 3.14. *Let φ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$ and $\varphi(1) = 1$. Assume also that φ is analytic on $[0, \varphi_{-1}(a)]$, where $0 < a < 1$, with $\varphi'(x) > 0$ for $0 \leqslant x \leqslant \varphi_{-1}(a)$. Then $\mathcal{F}_a$ is contained in $V_\varphi^\infty(\mathcal{C}_0[0,1])$ and $\overline{\lim}_{n\to\infty} \|V_\varphi^{-n} f\|_\infty^{1/n^2} \leqslant \sqrt{\gamma}$ for each $f \in \mathcal{F}_a$, where $\gamma = \max_{[0,\varphi^{-1}(a)]} 1/\varphi'$.*

Proof. We set $\psi = \varphi_{-1}$ for the inverse of φ. Clearly, ψ is analytic on $[0,a]$ and $\max_{[0,a]} |\psi'| = \gamma$. It is easy to check that

$$(Sf)(x) = \begin{cases} \dfrac{f'(\psi(x))}{\varphi'(\psi(x))} & \text{if } x \in (0,a), \\ 0 & \text{otherwise} \end{cases} \tag{3.26}$$

acts from $\mathcal{E}_a$ into $\mathcal{E}_a$ and that $V_\varphi S f = f$ for each f in $\mathcal{E}_a$. Therefore, it follows that $\mathcal{F}_a \subset \mathcal{E}_a \subset V_\varphi^\infty(\mathcal{C}_0[0,1])$ and the operator S defined in (3.26) coincides with the restriction to $\mathcal{E}_a$ of V_φ^{-1} acting on $V_\varphi^\infty(\mathcal{C}_0[0,1])$.

Now consider the operator T_ψ acting on $\mathcal{E}_a$ as defined in (3.18). One easily sees that $C_\psi S f = T_\psi C_\psi f$ for each f in $\mathcal{E}_a$, where C_ψ is defined as $(C_\psi f)(x) = f(\psi(x))$. Hence $C_\psi S^n f = T_\psi^n C_\psi f$ for each f in $\mathcal{E}_a$ and $n \geqslant 0$. Thus

$$\|V_\varphi^{-n} f\|_\infty = \|S^n f\|_\infty = \|C_\psi S^n f\|_\infty = \|T_\psi^n C_\psi f\|_\infty \quad \text{for } f \in \mathcal{E}_a \text{ and } n \geqslant 0. \tag{3.27}$$

Now, if f belongs to $\mathcal{F}_a$, then, by Lemma 3.13, we have that $C_\psi f$ belongs to $\mathcal{F}_a$. Hence $\overline{\lim}_{n\to\infty} (M_n(C_\psi f))^{1/n^2} \leqslant 1$. Applying Lemma 3.10 with $c_n = 1$ and $f_n = C_\psi f$ for each $n \geqslant 0$, we obtain $\overline{\lim}_{n\to\infty} \|T_\psi^n C_\psi f\|_\infty^{1/n^2} \leqslant \sqrt{\gamma}$. Therefore, using (3.27), we have $\overline{\lim}_{n\to\infty} \|V_\varphi^{-n} f\|_\infty^{1/n^2} \leqslant \sqrt{\gamma}$, which is the required conclusion. □

Now, we have all necessary tools to prove Theorem 3.6.

Proof of Theorem 3.6. One easily checks that F_b is linear and that $V_\varphi(F_b)$ and $V_\varphi^{-1}(F_b)$ are contained in F_b, which implies that $V_\varphi(F_b) = F_b = V_\varphi^{-1}(F_b)$. Thus we need only prove that F_b is dense in $\mathcal{C}_0[0,1]$.

Set $\psi = \varphi_{-1}$ for the inverse of φ. Since $b > 1/\varphi'(0)$, we may choose $0 < a < 1$ such that φ is analytic on $[0, \varphi_{-1}(a)]$ and $\varphi'(x) > 0$ for each $0 \leqslant x \leqslant \varphi_{-1}(a)$ and $\gamma = \max_{[0,\varphi_{-1}(a)]} 1/|\varphi'| = \max_{[0,a]} |\psi'| \leqslant b$. By Lemma 3.14, we have $\mathcal{F}_a \subset V_\varphi^\infty(\mathcal{C}_0[0,1])$ and $\overline{\lim}_{n\to\infty} \|V_\varphi^n f\|_\infty^{1/n^2} \leqslant \sqrt{\gamma} \leqslant \sqrt{b}$ for each $f \in \mathcal{F}_a$. Thus we find that $\mathcal{F}_a \subset F_b$. Since F_b is invariant under V_φ, we find that $\operatorname{span}\left(\bigcup_{n=0}^\infty V_\varphi^n(\mathcal{F}_a)\right) \subseteq F_b$. By Lemma 3.8, we know that $\mathcal{F}_a$ is dense in the subspace $\mathcal{C}_a$ of functions in $\mathcal{C}_0[0,1]$ that vanish on $[a,1]$. Hence, since V_φ is bounded, it follows that $\overline{\operatorname{span}}\left(\bigcup_{n=0}^\infty V_\varphi^n(\mathcal{C}_a)\right) \subseteq \overline{F}_b$, where the closures are taken in $\mathcal{C}_0[0,1]$. We may conclude from Lemma 3.12 that the left-hand side in the latter inclusion coincides with $\mathcal{C}_0[0,1]$ and, therefore, F_b is dense in $\mathcal{C}_0[0,1]$. The proof is complete. □

3.4. *Orbits of V_φ : lower estimate*

We begin with the following lemma that provides a lower estimate for orbits of V_φ.

Theorem 3.15. *Let φ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$ and analytic at 1 with $\varphi(1) = 1$. Then, for each non-zero f in $L^1[0,1]$, we have*

$$\varliminf_{n\to\infty} \|V_\varphi^n f\|_1^{1/n^2} \geqslant \frac{1}{\sqrt{\varphi'(1)}}. \tag{3.28}$$

Proof. Recall, from [**15**, § 2], that the adjoint $V_\varphi^\star$ that acts on $L^\infty[0,1]$ is $V_\varphi^\star = UV_\phi U$, where U is the involutive isometry defined by $(Ug)(x) = g(1-x)$ and $\phi(x) = 1 - \varphi_{-1}(1-x)$. Since φ is analytic at 1, so is $\psi = \phi^{-1}$ at 0. Take $\gamma > \psi'(0) \geqslant 1$ and let $0 < a < 1$ be such that ψ is analytic on $[0,a]$ and $\sup_{[0,a]} \psi' \leqslant \gamma$.

By Lemma 3.11, we have that $\sup\operatorname{supp}(V_\varphi^k g)$ tends to 1 as k tends to ∞ for each non-zero g in $L^1[0,1]$. Therefore, for each non-zero g in $L^1[0,1]$, we have $\sup\operatorname{supp}(V_\varphi^k g) > 1-a$ for all k large enough. Observe also that, for any $k \geqslant 1$ and for any non-zero g in $L^1[0,1]$, the inequality in (3.28) is satisfied for $f = g$ if and only if it is satisfied for $f = V_\varphi^k g$. Since the range of V_φ is contained in $\mathcal{C}_0[0,1]$, it is enough to show the inequality in (3.28) for each f in $\mathcal{C}_0[0,1]$ with $\sup\operatorname{supp}(f) > 1-a$.

Thus assume that f in $\mathcal{C}_0[0,1]$ has $\sup\operatorname{supp}(f) > 1-a$. We may take $1-a < b < 1$ and $\delta > 0$ such that $1-a < b-\delta < b+\delta \leqslant 1$ and $f(b) \neq 0$. By Lemma 3.8, $\mathcal{F}^+[1-b-\delta, 1-b+\delta]$ is dense in $\mathcal{C}_{00}^+[1-b-\delta, 1-b+\delta]$. In particular, there is g_1 in $\mathcal{F}[1-b-\delta, 1-b+\delta]$ such that $g_1(x) \geqslant 0$ for each $1-b-\delta \leqslant x \leqslant 1-b+\delta$ and

$$\int_{1-b-\delta}^{1-b+\delta} g_1(x)\,dx = 1.$$

We may think of g_1 as defined on the whole real line, by just making g_1 equal to 0 outside of $[1-b-\delta, 1-b+\delta]$. Now, consider $g_n(x) = ng_1(nx - (1-b)(n-1))$ for $n \geqslant 1$. In this way, $\{g_n\}_{n\geqslant 1}$ is a positive summability kernel at 0 (see [**12**, pp. 9–10]). Since $\operatorname{supp}(g_n) \subseteq [1-b-\delta/n, 1-b+\delta/n] \subset [0,a]$, we may regard $\{g_n\}$ as a sequence in $\mathcal{E}_a$. Now, set $f_n(x) = g_n(\psi(x))$ and consider $R = \sup_{n\geqslant 2} (M_n(\psi)/\gamma)^{1/(n-1)}$, where $M_n(\psi) = \sup_{[0,a]} |\psi^{(n)}|/n!$. By Lemma 3.9 we find that

$$M_n(f_k) \leqslant k \sum_{n=k_1+\ldots+nk_n} \frac{(k_1+\ldots+k_n)!}{k_1!\ldots k_n!} M_{k_1+\ldots+k_n}(g_1)(M_1(\psi))^{k_1}\ldots(M_n(\psi))^{k_n} k^{k_1+\ldots+k_n}.$$

Since $M_n(\psi) \leqslant \gamma R^{n-1}$, setting $\alpha_n = \max_{0\leqslant j\leqslant n} M_j(g_1)$, we have that

$$\begin{aligned} M_n(f_k) &\leqslant kR^n \sum_{n=k_1+\ldots+nk_n} \frac{(k_1+\ldots+k_n)!}{k_1!\ldots k_n!} M_{k_1+\ldots+k_n}(g_1)\left(\frac{k\gamma}{R}\right)^{k_1+\ldots+k_n} \\ &\leqslant kR^n\alpha_n \sum_{n=k_1+\ldots+nk_n} \frac{(k_1+\ldots+k_n)!}{k_1!\ldots k_n!}\left(\frac{k\gamma}{R}\right)^{k_1+\ldots+k_n}. \end{aligned}$$

Applying (3.20), we obtain $M_n(f_k) \leqslant \alpha_n k^2 R^{n-1}(1+k\gamma/R)^{n-1} = \alpha_n k^2(R+k\gamma)^{n-1} \leqslant \alpha_n(R+k\gamma)^{n+1}$. Since g_1 belongs to $F_{\psi(a)}$, we have $\overline{\lim}_{n\to\infty} \alpha_n^{1/n^2} \leqslant 1$ and, therefore, all the hypotheses of Lemma 3.10 with $c_k = R + k\gamma$ are fulfilled. Thus

$$\overline{\lim_{n\to\infty}} \|T_\psi^n f_n\|_\infty^{1/n^2} \leqslant \sqrt{\gamma}, \tag{3.29}$$

where T_ψ is as in (3.18). Since $V_\phi^{-1} f = C_\phi T_\psi C_\phi^{-1} f = C_\phi T_\psi C_\psi f$ for each $f \in \mathcal{E}_a$, we find that $g_n = V_\phi^n C_\phi T_\psi^n C_\psi g_n = V_\phi^n C_\phi T_\psi^n f_n$. Using that U is involutive and $V_\varphi^\star = UV_\phi U$, we see that $Ug_n = V_\varphi^{\star n} UC_\phi T_\psi^n f_n$. Therefore,

$$\|V_\varphi^n f\|_1 \geqslant \frac{|\langle V_\varphi^n f, UC_\phi T_\psi^n f_n\rangle|}{\|UC_\phi T_\psi^n f_n\|_\infty} \geqslant \frac{|\langle f, V_\varphi^{\star n} UC_\phi T_\psi^n f_n\rangle|}{\|UC_\phi T_\psi^n f_n\|_\infty} = \frac{|\langle f, Ug_n\rangle|}{\|T_\psi^n f_n\|_\infty}.$$

Since $\{g_n\}$ is a positive summability kernel at $1-b$, then so is $\{Ug_n\}$ at b. Therefore, $|\langle f, Ug_n\rangle|$ converges to $|f(b)| \neq 0$. Thus,

$$\|V_\varphi^n f\|_1 \geqslant \frac{|f(b)|}{\|T_\psi^n f_n\|_\infty}(1+o(1)) \quad \text{as } n \to \infty,$$

which along with (3.29) implies that $\underline{\lim}_{n\to\infty} \|V_\varphi^n f\|_p^{1/n^2} \geqslant 1/\sqrt{\gamma}$. Since $\gamma > \psi'(0) = \varphi'(1)$ was arbitrary, the result follows. □

From Lemma 3.15 and Corollary 3.4, we immediately have the following theorem.

Theorem 3.16. *Let φ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(1) = 1$ and $\varphi(x) < x$ for $0 < x < 1$. Assume also that φ is analytic at 1 and differentiable at 0 with $\varphi'(0) = 0$. Then, for each non-zero f in $L^p[0,1]$, $1 \leqslant p \leqslant \infty$, we have $\lim_{n\to\infty} \|V_\varphi f\|_p^{1/n^2} = 1/\sqrt{\varphi'(1)}$.*

4. *Dense generalized kernels*

In the next section, we will prove that if φ is continuous, strictly increasing and satisfies $\varphi(x) < x$ for $0 < x \leqslant 1$, then V_φ is supercyclic and $I + V_\varphi$ is hypercyclic when V_φ acts on $L^p[0,1]$, $1 \leqslant p < \infty$, or on $\mathcal{C}_0[0,1]$. To do this, we adopt a general point of view. We will show that if T is a continuous operator on a separable complete metrizable topological vector space X such that the span of the union of $\ker T^n \cap T^n(X)$ is dense in X, then the operator $I+T$ is hypercyclic. We will also show that in such a case T is supercyclic. This general point of view causes minimal extra effort and avoids the repetition of some arguments.

Recall that an *$\mathcal{F}$-space* is a complete metrizable topological vector space. The space of continuous linear operators on a topological vector space X is denoted by $\mathcal{L}(X)$.

Recall that a continuous operator T acting on a topological vector space X is said to be *hypercyclic* if there is x in X such that the orbit of x under T, that is, $\{T^n x\}_{n\geqslant 0}$, is dense in X and it is said to be *supercyclic* if there is x in X such that the projective orbit $\{\lambda T^n \text{ such that } \lambda \in \mathbb{C},\ n = 0,1,\ldots\}$ is dense in X. We say that T is *strongly hereditarily hypercyclic* if, for every subsequence $\{n_k\}$ of positive integers, there is x such that $\{T^{n_k}x\}$ is dense in X. Later concept has been used in [7]. Similarly, we can define *strongly hereditarily supercyclic* operators.

A bounded operator T acting on a locally convex topological vector space is called *weakly hypercyclic* or *weakly supercyclic* if it is hypercyclic or supercyclic with respect to the weak topology. Mazur's theorem asserts that the norm closure and the weak closure of convex sets coincide, and hence weakly supercyclic operators are cyclic. Hypercyclic and supercyclic operators have been intensely studied during the last few decades (see the surveys [9, 16] and references therein).

Let ℓ^p, $1 \leqslant p < \infty$, denote the Banach space of complex sequences that have p-summable modulus. Let $\{e_n\}_{n\geqslant 0}$ be the canonical basis of ℓ^p, where $1 \leqslant p < \infty$. Given a bounded sequence $\{w_n\}$ of non-zero complex numbers, the backward weighted shift with weight sequence $\{w_n\}$ is defined by $Te_0 = 0$ and $Te_n = w_n e_{n-1}$ for $n \geqslant 1$.

The next theorem, due to Salas [19], is one of the most important results on hypercyclicity for a fixed operator.

Theorem 4.1 (Salas' theorem). *Let T be a backward weighted shift on ℓ^2. Then the operator $I + T$ is hypercyclic.*

The next theorem extends Salas' theorem in several directions.

THEOREM 4.2. *Let T be a continuous operator on a separable $\mathcal{F}$-space X such that*

$$\ker^{\dagger} T = \operatorname{span}\left(\bigcup_{n=1}^{\infty}(T^n(X) \cap \ker T^n)\right)$$

is dense in X. Then $I+T$ is (*strongly hereditarily*) *hypercyclic.*

Recall that the *generalized kernel* of an operator T is the space $\ker^{\star} T = \bigcup_{n=1}^{\infty} \ker T^n$. It is worth mentioning that the space $\ker^{\dagger} T$ is contained in $T(X)$ as well as in $\ker^{\star} T$. Thus, any operator with dense $\ker^{\dagger} T$ has dense range and dense generalized kernel. Obviously, if T is a (unilateral) backward weighted shift on ℓ^p, then $\ker^{\star} T = \ker^{\dagger} T$ is the space of sequences with finite support, which is dense in ℓ^p, where $1 \leqslant p < \infty$. Hence Theorem 4.2 implies Salas' theorem. It is also worth noting that if $T(\ker T^{n+1})$ is dense in $\ker T^n$ for each positive integer n, then $\ker^{\dagger} T$ is dense in $\ker^{\star} T$. Thus, we have the following corollary.

COROLLARY 4.3. *Let T be a continuous operator on a separable $\mathcal{F}$-space X such that $\ker^{\star} T$ is dense in X and $T(\ker T^{n+1})$ is dense in $\ker T^n$ for each positive integer n. Then $I+T$ is* (*strongly hereditarily*) *hypercyclic.*

The advantage of the above corollary is that it is much easier to check that $T(\ker T^{n+1})$ is dense in $\ker T^n$.

A *generalized backward shift* is a continuous operator T on a topological vector space X such that $\ker T$ is one-dimensional and $\ker^{\star} T$ is dense in X. A dimension argument shows immediately that if T is a generalized backward shift, then $\ker T^n$ is n-dimensional and $T(\ker T^{n+1}) = \ker T^n$ for each positive integer n. From Corollary 4.3, we clearly have the following corollary.

COROLLARY 4.4. *Let X be a separable $\mathcal{F}$-space and T in $\mathcal{L}(X)$ be a generalized backward shift. Then $I+T$ is* (*strongly hereditarily*) *hypercyclic.*

REMARK. The fact that $I+T$ is hypercyclic for a generalized backward shift T on a separable $\mathcal{F}$-space also follows from Salas' theorem by means of a quasisimilarity argument, as already observed by several authors (see, for instance, [8]).

To prove Theorem 4.2, we need some preparation.

4.1. *A density criterion*

Recall that a topological space X is called a *Baire space* if, for each first category set $A \subset X$, its complement $X \setminus A$ is dense in X. According to the classical Baire theorem, complete metric spaces are Baire. We need the following easy proposition.

PROPOSITION 4.5. *Let X and Y be Baire topological spaces, where Y is second countable. Let $\{T_n\}_{n\geqslant 0}$ be a sequence of continuous maps from X to Y. Let Σ be the set of $(x,y) \in X \times Y$ for which there exists a sequence $\{x_n\}_{n\geqslant 0}$ in X such that $x_n \to x$ and $T_n x_n \to y$. If Σ is dense in $X \times Y$, then, for any subsequence $\{n_k\}_{k\geqslant 0}$, there is x such that $\{T_{n_k} x\}_{k\geqslant 0}$ is dense in X.*

Proof. Since Σ is dense in $X \times Y$, it is enough to apply Theorem 1 in [**9**, p. 348]. $\square$

4.2. *Invertible matrices*

To prove Theorem 4.2, we need to show that certain matrices are invertible. For each pair of positive integers n and k, consider the n-square matrix

$$M_{n,k} = \left(\frac{(k+n-l)!}{(k+n-l+j-1)!} \right)_{1\leqslant j,l\leqslant n}.$$

LEMMA 4.6. *For each pair n and k of positive integers, we have*

$$\det M_{n,k} = \frac{(n-1)!k!(k+1)!}{(k+n-1)!(k+n)!} \det M_{n-1,k+2}. \tag{4.1}$$

Proof. It is clear that (4.1) holds for $n = 2$. Thus suppose that $n \geqslant 3$. Subtracting from each column of $M_{n,k}$, except the first, the previous one, we see that

$$\det M_{n,k} = \det \begin{pmatrix} 1 & 0 & \dots & 0 \\ \dfrac{(k+n-1)!}{(k+n)!} & & & \\ \vdots & & N_{n,k} & \\ \dfrac{(k+n-1)!}{(k+2n-2)!} & & & \end{pmatrix},$$

where

$$N_{n,k} = \left(j \frac{(k+n-l-1)!}{(k+n-l+j)!} \right)_{1\leqslant j,l\leqslant n-1}.$$

Thus $\det M_{n,k} = \det N_{n,k}$. Now, dividing each jth row of $N_{n,k}$ by j and multiplying each lth column with $(k+n-l+1)!/(k+n-l-1)!$, we obtain $M_{n-1,k+2}$. Hence

$$\det M_{n,k} = \det M_{n-1,k+2} \prod_{\substack{j=1\\ l=1}}^{n-1} \frac{j(k+n-l-1)!}{(k+n-l+1)!} = \frac{(n-1)!k!(k+1)!}{(k+n-1)!(k+n)!} \det M_{n-1,k+2}.$$

The result is proved. $\square$

Consider now the n-square matrix $A_n = (1/(j+k-1)!)_{1\leqslant j,k\leqslant n}$. The key lemma in the proof of Salas' theorem is [**19**, Lemma 3.1], which asserts that A_n is invertible for $n = 2^k$ with k a positive integer. The latter is also used in [**13**] to prove that the operators in Salas' theorem do satisfy Kitai's criterion. Actually, A_n is invertible for each positive integer n. Indeed, $\det A_n$ can be computed explicitly.

LEMMA 4.7. *For each positive integer, the matrix A_n is invertible. Furthermore,* $\det A_1 = 1$, $\det A_2 = -1/12$ *and*

$$\det A_n = \frac{(-1)^{(n-1)n/2}}{(2n-1)!} \left(\prod_{j=1}^{2n-4} j! \right) \left(\prod_{j=n}^{2n-3} j!^{-2} \right) \quad \textit{for} \;\; n \geqslant 3.$$

Proof. Let B_n be the matrix obtained from A_n by reversing the order of the columns of A_n. Clearly, $\det A_n = (-1)^{(n-1)n/2} \det B_n$. Multiplying the jth column of B_n with $(n-j+1)!$

for $1 \leqslant j \leqslant n$, we obtain $M_{n,1}$. Hence, $\det A_n = (-1)^{(n-1)n/2} \det M_{n,1} \prod_{j=1}^n (j!)^{-1}$ for each $n \geqslant 1$. Now, the result follows by applying $n-1$ times (4.1) and then simplifying. □

Finally, for each pair of positive integers m and n with $m \geqslant 2n$, we consider the n-square matrix

$$B_{m,n} = \left(\binom{m}{k+j-1} \right)_{1 \leqslant j,k \leqslant n},$$

where $\binom{m}{k}$ denotes the binomial coefficient.

Lemma 4.8. *For each pair of positive integers m and n with $m \geqslant 2n$, we have that $B_{m,n}$ is invertible. Furthermore,* $\det B_{m,n} = \det A_n \prod_{j=-n}^{n} (m+j)^{n-|j|}$.

Proof. By multiplying the jth column of $B_{m,n}$ with $(m-j)!/m!$ for $1 \leqslant j \leqslant n$, we obtain $P_{m,n}$, whose entries are $p_{1,k} = 1/k!$, $1 \leqslant k \leqslant n$, and

$$p_{j,k} = \frac{(m-k)!}{(k+j-1)!(m-k-j+1)!} \quad \text{for } j \geqslant 2.$$

Consider $Q_{m,n}$ obtained from $P_{m,n}$ by replacing the jth row $P_{[j]}$ by $\sum_{l=0}^{j-1} \binom{j-1}{l} P_{[l+1]}$. Clearly, $\det P_{m,n} = \det Q_{m,n}$. In addition, one easily checks that the entries of $Q_{m,n}$ are $q_{1,k} = 1/k!$, $1 \leqslant k \leqslant n$, and

$$q_{j,k} = \frac{(m+j-1)!}{m!(k+j-1)!} \quad \text{for } j \geqslant 2.$$

Multiplying the jth row of $Q_{m,n}$ with $m!/(m+j-1)!$ for $2 \leqslant j \leqslant n$, we arrive at A_n. Upon putting everything together, we obtain

$$\det B_{m,n} = \left(\prod_{j=1}^{n} \frac{m!}{(m-j)!} \right) \left(\prod_{j=2}^{n} \frac{(m+j-1)!}{m!} \right) \det A_n.$$

Simplifying, the required formula for $\det B_{m,n}$ follows. □

4.3. *Proof of Theorem* 4.2

For x in $\mathbb{C}^n$, $n \geqslant 1$, we denote by x_j its jth coordinate.

Lemma 4.9. *Let S in $\mathcal{L}(\mathbb{C}^{2n})$, $n \geqslant 1$, be defined on the canonical basis $\{e_i : 1 \leqslant i \leqslant 2n\}$ by $Se_i = e_{i-1}$, $2 \leqslant i \leqslant 2n$ and $Se_1 = 0$. Then, for each $m \geqslant 2n$ and each u and v in $\mathbb{C}^n$, there exists a unique $x = x(m)$ in $\mathbb{C}^{2n}$ such that:*
(a) *$x_j(m) = u_j$, for $1 \leqslant j \leqslant n$;*
(b) *$((I+S)^m x(m))_j = v_j$, for $1 \leqslant j \leqslant n$.*
Furthermore,

$$|x_{n+j}(m)| = O(m^{-j}) \quad \text{as } m \to \infty \text{ for } 1 \leqslant j \leqslant n, \tag{4.2}$$

$$|((I+S)^m x(m))_{n+j}| = O(m^{-j}) \quad \text{as } m \longrightarrow \infty \text{ for } 1 \leqslant j \leqslant n. \tag{4.3}$$

Proof. For y in $\mathbb{C}^{2n}$ and z in $\mathbb{C}^n$, we define $\widetilde{y} = (y_{n+1}, \dots, y_{2n}) \in \mathbb{C}^n$ and $\widehat{z} = (z_1, \dots, z_n, 0, \dots, 0)$ in $\mathbb{C}^{2n}$. Let also $w(m)$ in $\mathbb{C}^n$ be defined by $w_j(m) = v_{n-j+1} - ((I+S)^m \widehat{u})_{n-j+1}$ for $1 \leqslant j \leqslant n$. One easily sees that there is a unique $x(m)$ satisfying (a) and (b) if and only if the equation

$$B_{m,n} \widetilde{x} = w(m), \tag{4.4}$$

where $B_{m,n}$ is the matrix defined in the previous subsection, has a unique solution. Thus the first statement of the lemma follows from Lemma 4.8.

It remains to show that (4.2) and (4.3) also hold. To this end, first observe that $w_j(m) = v_{n-j+1} - \sum_{l=0}^{j-1} \binom{m}{l} u_{n-j+1+l}$ for $1 \leqslant j \leqslant n$. Thus

$$w_j^m = O(m^{j-1}) \quad \text{as } m \longrightarrow \infty \text{ for } 1 \leqslant j \leqslant n. \tag{4.5}$$

Now consider the n-diagonal matrix $D_{m,n}$ with entries m^{j-1}, $1 \leqslant j \leqslant n-1$, in the main diagonal. An easy computation shows that $B_{m,n} = mD_{m,n}C_{m,n}D_{m,n}$, where $C_{m,n} = \{\gamma_{j,k}\}_{1\leqslant j,k\leqslant n}$ has entries $\gamma_{1,1} = 1$ and $\gamma_{j,k} = (1/(j+k-1)!)\prod_{l=1}^{j+k-2}(1-l/m)$ for $(j,k) \neq (1,1)$. Since $B_{m,n}$ as well as $D_{m,n}$ are invertible, so is $C_{m,n}$ and (4.4) implies that $\widetilde{x}^m = B_{m,n}^{-1} w(m) = m^{-1} D_{m,n}^{-1} C_{m,n}^{-1} D_{m,n}^{-1} w(m)$.

From (4.5), the sequence $\{D_{m,n}^{-1} w(m)\}_{m\geqslant 2n}$ is bounded in $\mathbb{C}^n$. On the other hand, the sequence of invertible matrices $\{C_{m,n}\}_{m\geqslant 2n}$ converges to the matrix A_n defined in Subsection 4.2, which is invertible by Lemma 4.7. Hence, the sequence $\{C_{m,n}^{-1}\}$ converges to A_n^{-1} as m tends to ∞ and, therefore, the sequence $\{C_{m,n}^{-1} D_{m,n}^{-1} w^m\}_{m\geqslant 2n}$ is bounded in $\mathbb{C}^n$. Hence,

$$x_{n+j}(m) = \widetilde{x}_j(m) = m^{-1}(D_{m,n}^{-1} C_{m,n}^{-1} D_{m,n}^{-1} w^m)_j, \quad 1 \leqslant j \leqslant n,$$

satisfy (4.2). Finally, since $((I+S)^m x(m))_{n+j} = \sum_{l=0}^{n-j} \binom{m}{l} x_{n+j+l}(m)$ for $1 \leqslant j \leqslant n$, the estimates in (4.3) follow from (4.2) and the result is proved. □

Lemma 4.9 allows us to prove the following lemma.

Lemma 4.10. *Let T be a continuous operator on a topological vector space X. Assume that x belongs to $T^m(X) \cap \ker T^m$, where m is a positive integer. Then there exist sequences $\{u_k\}_{k\geqslant 0}$ and $\{v_k\}_{k\geqslant 0}$ in X such that*

$$u_k \longrightarrow 0, \ (I+T)^k u_k \longrightarrow x, \ v_k \longrightarrow x \quad \text{and} \quad (I+T)^k v_k \longrightarrow 0 \ \text{as } n \longrightarrow \infty. \tag{4.6}$$

Proof. If $x = 0$, then it is enough to take $u_k = v_k = 0$. Thus, assume that $x \neq 0$. We will show that the proof reduces to the operator S defined in Lemma 4.9. Let n be the smallest positive integer for which $T^n x = 0$. In particular, $n \leqslant m$, which implies that x belongs to $T^n(X)$. Thus we may choose w in X such that $T^n w = x$. We set $h_j = T^{2n-j} w$ for $1 \leqslant j \leqslant 2n$ and $Y = \operatorname{span}\{h_1, \ldots, h_{2n}\}$. In particular, we have $Th_j = h_{j-1}$, $2 \leqslant j \leqslant 2n$, and $Th_1 = T^{2n} h_{2n} = T^n x = 0$. Thus clearly, Y is invariant under T. Since $h_1 = T^{2n-1} h_{2n} = T^{n-1} x \neq 0$, it follows that $\dim Y \geqslant 2n$ and, therefore, $\{h_1, \ldots, h_{2n}\}$ is a basis of Y.

Let J be the operator from $\mathbb{C}^{2n}$ onto Y defined by $Je_k = h_k$, $1 \leqslant k \leqslant 2n$. Clearly, T acting on Y is similar under J to S acting on $\mathbb{C}^{2n}$, where S is the operator defined on Lemma 4.9. Now, $J^{-1} x = e_n$. Thus taking, $u = (0, \ldots, 0, 1)$ in $\mathbb{C}^n$ and $v = (0, \ldots, 0)$, we find that there is a sequence $\{g_k\}_{k\geqslant 0}$ in $\mathbb{C}^{2n}$ such that $g_k \to e_n$ and $(I+S)^k g_k \to 0$ as $k \to \infty$. Applying Lemma 4.9 with $u = (0, \ldots, 0, 0)$ and $v = (0, \ldots, 0, 1)$, we find that there is a sequence $\{f_k\}_{k\geqslant 0}$ in $\mathbb{C}^{2n}$ such that $f_k \to 0$ and $(I+S)^k f_k \to e_m$ as $k \to \infty$. The result follows because two topological vector spaces of the same finite dimension are homeomorphic under any algebraic isomorphism between them. □

Lemma 4.11. *Let T be a continuous operator on a topological vector space X. Assume that x and y belong to $\ker^{\dagger} T$. Then there exists a sequence $\{x_k\}$ in X such that $x_k \to x$ and $(I+T)x_k \to y$ as $k \to \infty$.*

Proof. Let Σ be the set of (x, y) in $X \times X$ for which there is a sequence $\{x_n\}$ in X such that x_n tends to x and $(I+T)^n x_n$ tends to y. By Lemma 4.10, we have $\ker T^n \cap T^n(X) \times \{0\} \subset \Sigma$ and $\{0\} \times \ker T^n \cap T^n(X) \subset \Sigma$ for each $n \geqslant 1$. On the other hand, it is clear that Σ is a subspace of $X \times X$. Therefore, one immediately obtains that $\ker^\dagger T \times \ker^\dagger T \subseteq \Sigma$, which is what had to be proved. □

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. Let Σ be the set of $(x, y) \in X \times X$ for which there is $\{x_n\}$ in X such that $x_n \to x$ and $(I+T)^n x_n \to y$. By Lemma 4.11, it follows that Σ contains $\ker^\dagger T \times \ker^\dagger T$. Since $\ker^\dagger T$ is dense in X, we obtain that Σ is dense in $X \times X$. According to Theorem 4.5, for each subsequence $\{n_k\}$ there is x in X such that $\{(I+T)^{n_k} x\}$ is dense in X, that is, $I+T$ is strongly hereditarily hypercyclic. The proof of Theorem 4.2 is complete. □

4.4. *Supercyclicity*

To prove the supercyclicity of V_φ, we extend another result by Salas [**20**].

Proposition 4.12. *Let X be a separable $\mathcal{F}$-space and T be in $\mathcal{L}(X)$. Assume also that T has dense range and dense generalized kernel. Then T is* (*strongly hereditarily*) *supercyclic.*

The advantage of Proposition 4.12 over Corollary 2.8 in [**20**] is that we avoid the existence of the local inverse.

The next criterion for an operator to be strongly hereditarily supercyclic is analogous to one of the forms of the Supercyclicity Criterion (see [**16**]).

Theorem 4.13. *Let T be a continuous operator on an $\mathcal{F}$-space X and $\{\lambda_k\}_{k\geqslant 0}$ be a sequence of non-zero complex numbers. Assume also that there exist dense subsets E and F of X and mappings $S_k : F \to X$ such that $T^k S_k y \to y$ and $\lambda_k^{-1} S_k y \to 0$ for each $y \in F$ and $\lambda_k T^k x \to 0$ for each $x \in E$ as $k \to \infty$. Then, for any strictly increasing sequence $\{n_k\}_{k\geqslant 0}$ of positive integers, there exists $x \in X$ for which $\{\lambda_{n_k} T^{n_k} x : k \geqslant 0\}$ is dense in X.*

Proof of Proposition 4.12. Let d be a metric that induces the topology of X. Let F be a dense countable subset of X. Since T has dense range, we find that $T^k(X)$ is dense in X for each $k \geqslant 0$. Hence, we may choose $S_k : F \to X$ such that $d(y, T^k S_k y) < 2^{-k}$ for each y in F and each $k \geqslant 0$. Clearly, $T^k S_k y \to y$ for each y in F. Since F is countable and X is metrizable, there is a sequence $\{\lambda_n\}$ of positive numbers such that $\lambda_n^{-1} S_n y \to 0$ as $n \to \infty$ for each y in F. Finally, $E = \ker^\star T$ is dense in X and, for each y in E, we have $T^n y = 0$ for all n large enough and, therefore, $\lambda_n T^n y \to 0$ as $n \to \infty$. Thus all the hypotheses of Theorem 4.13 are fulfilled and we conclude that T is strongly hereditarily supercyclic. □

5. *Supercyclicity of V_φ and hypercyclicity $I + V_\varphi$*

In this section we study the supercyclicity of V_φ as well as the hypercyclicity of $I + V_\varphi$ acting on the spaces $L^p[0, 1]$, $1 \leqslant p < \infty$. Since V_φ is a contraction on $L^p[0, 1]$, $1 \leqslant p < \infty$, it cannot be weakly hypercyclic.

The following easy proposition states that if V_φ is weakly supercyclic, then $\varphi(x) \leqslant x$ almost everywhere.

PROPOSITION 5.1. *Let φ is a measurable self-map of $[0,1]$ with $\varphi(x) > x$ on a set of positive Lebesgue measure. Then V_φ acting on $L^p[0,1]$, $1 \leqslant p < \infty$, is not weakly supercyclic.*

Proof. A supercyclic compact operator on a Banach space must be quasinilpotent (see [**10**]) and the same is true for weakly supercyclic operators (the same argument works). By [**15**, Corollary 2.2], the operator V_φ is not quasinilpotent and the result follows. □

In what follows, we will be considering only continuous symbols. The following lemmas describe the closure of the range of V_φ. We denote by $\overline{\operatorname{ran}}_p V_\varphi$ the closure of the range of V_φ acting on $L^p[0,1]$ and, when acting on $\mathcal{C}[0,1]$ or $\mathcal{C}_0[0,1]$, it will be denoted by $\overline{\operatorname{ran}}\, V_\varphi$ or $\overline{\operatorname{ran}}_0 V_\varphi$, respectively.

LEMMA 5.2. *Let φ be a continuous self-map of $[0,1]$. Assume that V_φ acts on $\mathcal{C}[0,1]$. If φ is not strictly monotone, then the codimension of $\overline{\operatorname{ran}}\, V_\varphi$ is infinite. If φ is strictly monotone and $\varphi(0) \neq 0$, $\varphi(1) \neq 0$, then $\overline{\operatorname{ran}}\, V_\varphi = \mathcal{C}[0,1]$. If φ is strictly monotone and $\varphi(0) = 0$, then $\overline{\operatorname{ran}}\, V_\varphi = \{f \in \mathcal{C}[0,1] : f(0) = 0\}$. Finally, if φ is strictly monotone and $\varphi(1) = 0$, then $\overline{\operatorname{ran}}\, V_\varphi = \{f \in \mathcal{C}[0,1] : f(1) = 0\}$.*

Proof. If φ is not strictly monotone, then $A = \{(t,s) \in [0,1]^2 : t < s \text{ and } \varphi(t) = \varphi(s)\}$ is infinite. Since $\overline{\operatorname{ran}}\, V_\varphi \subseteq \{f \in \mathcal{C}[0,1] : f(t) = f(s) \text{ for each } (t,s) \in A\}$ and the last space has infinite codimension, $\overline{\operatorname{ran}}\, V_\varphi$ has infinite codimension in $\mathcal{C}[0,1]$.

The description of $\overline{\operatorname{ran}}\, V_\varphi$ in the case when φ is strictly monotone follows from the decomposition $V_\varphi = C_\varphi V$ and the fact that the closure of the range of the Volterra operator acting on $\mathcal{C}[0,1]$ is $\mathcal{C}_0[0,1]$. Indeed, if $\varphi(0) \neq 0$ and $\varphi(1) \neq 0$, then $C_\varphi(\mathcal{C}_0[0,1]) = \mathcal{C}[0,1]$; if $\varphi(0) = 0$, then $C_\varphi(\mathcal{C}_0[0,1]) = \mathcal{C}_0[0,1]$; and finally if $\varphi(1) = 0$, then $C_\varphi(\mathcal{C}_0[0,1]) = \{f \in \mathcal{C}[0,1] : f(1) = 0\}$. □

LEMMA 5.3. *Let φ be a continuous self-map of $[0,1]$. Assume that V_φ acts on $L^p[0,1]$ with $1 \leqslant p < \infty$. If φ is not strictly monotone, then $\overline{\operatorname{ran}}_p V_\varphi$ has infinite codimension. If φ is strictly monotone, then $\overline{\operatorname{ran}}_p V_\varphi = L^p[0,1]$.*

Proof. One can easily verify that $\overline{\operatorname{ran}}_p V_\varphi \cap \mathcal{C}[0,1] = \overline{\operatorname{ran}}\, V_\varphi$. Thus the result follows immediately from the previous lemma and the fact that both $\mathcal{C}_0[0,1]$ and $\{f \in \mathcal{C}[0,1] : f(1) = 0\}$ are dense in $L^p[0,1]$. □

The following lemma is an immediate consequence of Lemma 5.2.

LEMMA 5.4. *Let φ be a continuous self-map of $[0,1]$ satisfying $\varphi(0) = 0$. Assume that V_φ acts on $\mathcal{C}_0[0,1]$. If φ is not strictly increasing, then $\overline{\operatorname{ran}}_0 V_\varphi$ has infinite codimension. If φ is strictly increasing, then $\overline{\operatorname{ran}}_0 V_\varphi = \mathcal{C}_0[0,1]$.*

Now, we can show that the cyclicity of V_φ is a severe restriction on φ.

PROPOSITION 5.5. *Let φ be a continuous self-map of $[0,1]$. Assume that V_φ acting on $L^p[0,1]$, $1 \leqslant p < \infty$, or on $\mathcal{C}[0,1]$ is cyclic. Then φ is strictly monotone. In addition, if $\varphi(0) = 0$ and V_φ is cyclic when acting on $\mathcal{C}_0[0,1]$, then φ is strictly increasing.*

Proof. It is known and easy to see that the closure of the range of a cyclic operator is at most of codimension 1. Thus it remains to apply Lemmas 5.2–5.4. □

Since weakly supercyclic operators are cyclic, as a consequence of Propositions 5.1 and 5.5, we have the following corollary.

Corollary 5.6. *Let φ be a continuous self-map of $[0,1]$. If V_φ acting on $L^p[0,1]$, $1 \leqslant p < \infty$, or on $\mathcal{C}_0[0,1]$ is weakly supercyclic, then φ is strictly increasing and $\varphi(x) \leqslant x$ for $0 \leqslant x \leqslant 1$.*

The cyclic properties that we shall be considering are cyclic, weakly supercyclic, weakly hypercyclic, supercyclic and hypercyclic. Actually, the real core of the question, whether a Volterra composition operator satisfies any of these properties or not, is in the friendly Hilbert space setting $L^2[0,1]$.

Proposition 5.7. *Let φ be a continuous self-map of $[0,1]$ with $\varphi(0) > 0$ and $\varphi(1) > 0$. Then V_φ acting on $L^2[0,1]$ has a given cyclic property if and only if V_φ acting on $L^p[0,1]$, $1 \leqslant p < \infty$, or on $\mathcal{C}[0,1]$ has the same cyclic property.*

Proof. Let $1 < p < \infty$. First, observe that $\mathcal{C}[0,1]$ is densely and continuously embedded into $L^p[0,1]$, and the latter space is densely and continuously embedded into $L^1[0,1]$. The same holds true if all the spaces carry their weak topologies. Thus it suffices to show that if V_φ acting on $L^1[0,1]$ has a cyclic property, then so does V_φ acting on $\mathcal{C}[0,1]$.

Suppose that V_φ acting on $L^1[0,1]$ has a given cyclic property. By Proposition 5.5, φ is strictly monotone. By Lemma 5.2, V_φ acting on $\mathcal{C}[0,1]$ has dense range. Thus V_φ is a bounded linear operator from $L^1[0,1]$ to $\mathcal{C}[0,1]$ with dense range. Therefore, we find that if f in $L^1[0,1]$ provides a given cyclic property for V_φ acting on $L^1[0,1]$, then $V_\varphi f$ provides the same property for V_φ acting on $\mathcal{C}[0,1]$. □

The proof of the next proposition is similar to the one of Proposition 5.7 and we omit it. One has to use Lemma 5.4 instead of Lemma 5.2.

Proposition 5.8. *Let φ be a continuous self-map of $[0,1]$ with $\varphi(0) = 0$. Then V_φ acting on $L^2[0,1]$ has a given cyclic property if and only if V_φ acting on $L^p[0,1]$, $1 \leqslant p < \infty$, or on $\mathcal{C}_0[0,1]$ has the same cyclic property.*

5.1. *Supercyclicity of V_φ and hypercyclicity of $I + V_\varphi$: case $\varphi(1) < 1$*

Although V_φ acting on $L^2[0,1]$ cannot be weakly hypercyclic, we have the following theorem.

Theorem 5.9. *Let φ be a continuous strictly increasing self-map of $[0,1]$ such that $\varphi(x) < x$ for $0 < x \leqslant 1$. Then V_φ is supercyclic and $I + V_\varphi$ is hypercyclic.*

Proof. Clearly, the sequence $\{\varphi_n(1)\}$ is strictly decreasing and tends to 0 as n tends to ∞. One can easily verify that $\ker V_\varphi^n = \{f \text{ such that } \inf \operatorname{supp}(f) \geqslant \varphi_n(1)\}$. Now, a straightforward argument shows that $V_\varphi(\ker V_\varphi^{n+1})$ is dense in $\ker V_\varphi^n$ for each positive integer n and that $\ker^\star V_\varphi$ is dense in the underlying space. Therefore, the result follows from Proposition 4.12 and Corollary 4.3. □

From Corollary 5.6, it follows that φ cannot fail to be strictly increasing or to have the graph below the identity function. However, $\varphi(1) < 1$ is a different issue.

5.2. *Supercyclicity of V_φ: case $\varphi(1) = 1$*

Although the Volterra operator is not weakly supercyclic (see [**17**]) there are supercyclic Volterra composition operators whose symbols are below the diagonal and take the value 1 at 1. Along this subsection we will prove the following theorem.

THEOREM 5.10. *Let φ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$ and $\varphi(1) = 1$ and analytic at 0. If $\varphi'(0) > \delta_1^+$, where*

$$\delta_1^+ = \delta_1^+(\varphi) = \overline{\lim_{x\to 1}} \frac{1-x}{1-\varphi(x)},$$

then V_φ is supercyclic. In particular, if φ is differentiable at 1 and $\varphi'(0)\varphi'(1) > 1$, then V_φ is supercyclic.

Proof. By Proposition 5.8, it is enough to show that V_φ is supercyclic on $\mathcal{C}_0[0,1]$. We take $b > 0$ with $1/\varphi'(0) < b < 1/\delta_1^+$ and consider the dense subspace of $\mathcal{C}_0[0,1]$ defined by $E = \{f \in \mathcal{C}_0[0,1] : \inf \operatorname{supp}(f) > 0\}$. According to Lemma 3.5, we have

$$\overline{\lim_{n\to\infty}} \|V_\varphi^n f\|_\infty^{1/n^2} \leqslant \sqrt{\delta_1^+} \quad \text{for each } f \in E. \tag{5.1}$$

On the other hand, by Theorem 3.6, $F = \{f \in V_\varphi^\infty(\mathcal{C}_0[0,1]) : \overline{\lim}_{n\to\infty} \|V_\varphi^{-n} f\|_\infty^{1/n^2} \leqslant \sqrt{b}\}$ is a dense linear subspace of $\mathcal{C}_0[0,1]$ satisfying $V_\varphi(F) = F = V_\varphi^{-1}(F)$. Let S be the restriction of V_φ^{-1} to F. Clearly, $V_\varphi S f = f$ for each f in F and

$$\overline{\lim_{n\to\infty}} \|S^n f\|_\infty^{1/n^2} \leqslant \sqrt{b} \quad \text{for each } f \in F. \tag{5.2}$$

Finally, take $b < c < 1/\delta_1^+$ and let $\lambda_n = c^{n^2/2}$ for $n \geqslant 0$. Inequalities (5.2) and (5.1) imply that $\lambda_n V_\varphi^n f$ tends to 0 as n tends to ∞ for each f in E and $\lambda_n^{-1} S^n f$ tends to 0 for each f in F. Upon applying Theorem 4.13 with $T = V_\varphi$ and $S_k = S^k$, we conclude that V_φ acting on $\mathcal{C}_0[0,1]$ is supercyclic. □

5.3. *Non-cyclicity*

The next theorem complements Theorem 5.10.

THEOREM 5.11. *Let φ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$ and $\varphi(1) = 1$ and analytic at 1. If $\varphi'(1)\delta_0^+ < 1$, where*

$$\delta_0^+ = \delta_0^+(\varphi) = \overline{\lim_{x\to 0}} \frac{\varphi(x)}{x},$$

then V_φ is not cyclic. In particular, if φ is differentiable at 0 with $\varphi'(0)\varphi'(1) < 1$, then V_φ is not cyclic.

Proof. By Proposition 5.8, it is enough to prove that V_φ is not cyclic on $L^2[0,1]$. Clearly, $\phi(x) = 1 - \varphi_{-1}(1-x)$ is continuous, strictly increasing, analytic at 0, $\phi(x) < x$ for $0 < x < 1$, $\phi(1) = 1$, $\phi'(0) = 1/\varphi'(1)$ and

$$\delta_1^+(\phi) = \overline{\lim_{x\to 1}} \frac{1-x}{1-\phi(x)} = \delta_0^+(\varphi).$$

In addition, the fact that $\varphi'(1)\delta_0^+(\varphi) < 1$ implies $\phi'(0) > \delta_1^+(\phi)$. Thus we may choose $1 \leqslant 1/\phi'(0) < b < 1/\delta_1^+(\phi)$. Since ϕ is analytic at zero, there is $0 < a < 1$ such that ϕ is analytic on $[0, \phi^{-1}(a)]$ and $\max_{[0,\phi^{-1}(a)]} 1/\phi' \leqslant b$. For each n in $\mathbb{Z}$, we set $a_n = \phi_{-n}(a)$. We choose $a_{-1} < c < a_0$ and set $c_n = \phi_{-n}(c)$ for each n in $\mathbb{Z}$. Clearly, $\{a_n\}$ and $\{c_n\}$ converge to 1 as n tends to $+\infty$ and to 0 as n tends to $-\infty$. Moreover, $c_n < a_n < c_{n+1}$ for each n in $\mathbb{Z}$. By Lemma 3.8, there are non-zero functions f_0 in $\mathcal{F}[c_0, a_0]$ and f_1 in $\mathcal{F}[a_{-1}, c_0]$ that we extend to the whole interval $[0, 1]$ by defining them as zero outside their intervals of definition. By Lemma 3.14, we find that f_0 as well as f_1 are in $V_\phi^\infty(\mathcal{C}_0[0, 1])$, which we defined in Subsection 8.3, and

$$\varlimsup_{n\to\infty} \|V_\phi^{-n} f_j\|_2^{1/n^2} \leqslant \sqrt{b} \quad \text{for } j = 0, 1. \tag{5.3}$$

On the other hand, Lemma 3.5 implies that

$$\varlimsup_{n\to\infty} \|V_\phi^{n} f_j\|_2^{1/n^2} \leqslant \sqrt{\delta_1^+(\phi)} \quad \text{for } j = 0, 1. \tag{5.4}$$

Now, take real numbers $b < \alpha < \beta < 1/\delta_1^+(\phi)$ and set

$$z_n = \begin{cases} \alpha^{n(1-n)/2} & \text{if } n < 0, \\ \beta^{n(n+1)/2} & \text{if } n \geqslant 0. \end{cases}$$

From (5.3) and (5.4), it follows that $J(x \oplus y) = \sum_{n=-\infty}^{\infty} z_n(x_n V_\phi^n f_0 + y_n V_\phi^n f_1)$ defines a bounded operator from $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ into $L^2[0, 1]$. We need to show that $J^\star$ has dense range. To this end, it is enough to check that J is one-to-one. Let x and y be in $\ell^2(\mathbb{Z})$ and suppose that $J(x \oplus y) = 0$. By Lemma 3.11, it follows that $\inf\operatorname{supp}(V_\phi^n f_0) = c_n$ and $\inf\operatorname{supp}(V_\phi^n f_1) = a_{n-1}$ for each n in $\mathbb{Z}$ and $\sup\operatorname{supp}(V_\phi^n f_0) = a_n$ and $\sup\operatorname{supp}(V_\phi^n f_1) = c_n$ for $n \leqslant 0$. Thus, for each $n \leqslant 0$, we find that $V_\phi^n f_0$ is different from zero and supported on $[c_n, a_n]$ and, for each $m \neq n$, we have that $V_\phi^m f_j$ vanishes on $[c_n, a_n]$. Similarly, for each $n \leqslant 0$, we find that $V_\phi^n f_1$ is different from zero and supported in $[a_{n-1}, c_n]$ and, for each $m \neq n$, we have that $V_\phi^m f_j$ vanishes on $[a_{n-1}, c_n]$. It follows that $x_n = y_n = 0$ for $n \leqslant 0$. If $x \oplus y$ is different from zero, let n be the minimal positive integer for which $|x_n| + |y_n| > 0$. Since all $V_\phi^m f_j$ vanish on $[a_{n-1}, c_n]$, except for $m = n$ and $j = 1$, it follows that $y_n = 0$. Similarly, $x_n = 0$, which is a contradiction. Therefore, J is one-to-one.

Let $\{e_n\}_{n\in\mathbb{Z}}$ denote the canonical basis of $\ell^2(\mathbb{Z})$ and consider the (forward) weighted shift $Se_n = w_{n+1}e_{n+1}$, with weight sequence

$$w_n = \frac{z_{n-1}}{z_n} = \begin{cases} \alpha^{n-1} & \text{for } n \leqslant 0, \\ \beta^{-n} & \text{for } n \geqslant 1. \end{cases}$$

We have $V_\phi J = J(S \oplus S)$. Therefore, $J^\star V_\phi^\star = (S^\star \oplus S^\star)J^\star$. In [15, § 2], it is proved that $V_\phi^\star$ is unitarily similar under $(Uf)(x) = f(1 - x)$ to V_φ. Thus assuming that V_φ is cyclic, then so is $V_\phi^\star$. Let f in $L^2[0, 1]$ be cyclic for $V_\phi^\star$. Then

$$\operatorname{span}\{(S^{\star n} \oplus S^{\star n})(J^\star f) : n \geqslant 0\} = J^\star(\operatorname{span}\{V_\phi^{\star n} f : n \geqslant 0\}).$$

Since $J^\star$ has dense range, it follows that $J^\star f$ is cyclic for $S^\star \oplus S^\star$. Now, the operator R on $\ell^2(\mathbb{Z})$, defined by $Re_n = (\alpha/\beta)^{|n(n+1)|/2} e_{-n}$, n in $\mathbb{Z}$, is bounded because $\alpha < \beta$. One easily checks that $SR = RS^\star$. Hence, $(I \oplus R)(S^\star \oplus S^\star) = (S^\star \oplus S)(I \oplus R)$. Therefore,

$$\operatorname{span}\{(S^\star \oplus S)^n (I \oplus R)(J^\star f) : n \geqslant 0\} = (I \oplus R)(\operatorname{span}\{(S^\star \oplus S^\star)^n (J^\star f) : n \geqslant 0\}).$$

Taking into account that $J^\star f$ is cyclic for $S^\star \oplus S^\star$ and $I \oplus R$ has dense range, we see that $S^\star \oplus S$ is cyclic. Let $x \oplus y$ in $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ be cyclic for $S^\star \oplus S$ and consider the dual pairing

$$\langle u, v\rangle = \sum_{n\in\mathbb{Z}} u_n v_n, \quad u, v \in \ell^2(\mathbb{Z}).$$

Since $x \oplus y$ must be different from zero, the functional $\Phi(u \oplus v) = \langle u, y\rangle - \langle v, x\rangle$ on $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ is non-zero. However, for each $n \geqslant 0$, we have $\Phi((S^\star \oplus S)^n(x \oplus y)) = \langle S^{\star n}x, y\rangle - \langle S^n y, x\rangle = 0$, which contradicts the fact that $x \oplus y$ is cyclic for $S^\star \oplus S$. The proof is complete. □

5.4. *Hypercyclicity and supercyclicity of V_φ on $\mathcal{C}_0[0,1)$*

Let $\mathcal{C}_0[0,1)$ be the Fréchet space of continuous functions vanishing at 0 endowed with the topology of uniform convergence on compact subsets of $[0,1)$. Herzog and Weber [**11**] showed that V_φ, where $\varphi(x) = x^b$ with $0 < b < 1$, acting on $\mathcal{C}_0[0,1)$ is hypercyclic. Here, we shall provide an easy characterization in terms of the symbols φ of the hypercyclicity of V_φ acting on $\mathcal{C}_0[0,1)$. First, note that V_φ acts from $\mathcal{C}_0[0,1)$ into itself if and only if φ is a continuous map from $[0,1)$ into itself and $\varphi(0) = 0$.

PROPOSITION 5.12. *Let φ be a continuous self-map of $[0,1)$ with $\varphi(0) = 0$.*

(i) *If V_φ is weakly supercyclic on $\mathcal{C}_0[0,1)$, then φ is strictly increasing.*

(ii) *If φ is strictly increasing on $[0,1)$ and there are $0 < a < b < 1$ such that $\varphi(a) > a$ and $\varphi(b) = b$, then V_φ is not weakly supercyclic on $\mathcal{C}_0[0,1)$.*

(iii) *If φ is strictly increasing on $[0,1)$ and there is $0 < b < 1$ such with $\varphi(b) = b$, then V_φ is not weakly hypercyclic on $\mathcal{C}_0[0,1)$.*

Proof. To prove (i), observe that $\varphi(a) = \varphi(b)$ with $0 < a < b < 1$, then $(V_\varphi f)(a) = (V_\varphi f)(b)$ for each f in $\mathcal{C}_0[0,1)$. Thus the range of V_φ acting on $\mathcal{C}_0[0,1)$ is not dense and, therefore, V_φ is not weakly supercyclic on $\mathcal{C}_0[0,1)$.

To prove (ii) and (iii), first observe that since φ is increasing and $\varphi(b) = b$, we find that φ is also a self-map of $[0,b]$. Let P from $\mathcal{C}_0[0,1)$ onto $\mathcal{C}_0[0,b]$ be defined by $Pf = f|_{[0,b]}$. Set $\psi = \varphi|_{[0,b]}$ and consider V_ψ acting on $\mathcal{C}_0[0,b]$. Since $\varphi([0,b]) = [0,b]$, we see that $PV_\varphi^n f = V_\psi^n Pf$ for each f in $\mathcal{C}_0[0,1)$. Thus weak hypercyclicity (weak supercyclicity) of V_φ implies weak hypercyclicity (weak supercyclicity) of V_ψ. On the other hand, since $\|V_\psi\| < 1$, we find that V_ψ cannot be weakly hypercyclic on $\mathcal{C}_0[0,b]$ and, therefore, neither can V_φ acting on $\mathcal{C}_0[0,1)$. Finally, assume that there is $0 < a < b$ for which $\varphi(a) = \psi(a) > a$. Then the operator V_ψ cannot be weakly supercyclic because of Corollary 5.6 and, therefore, neither can V_φ acting on $\mathcal{C}_0[0,1)$. □

We also need a lemma, which is a particular case of Theorem 3.2.5 in [**23**], dealing with projective limits of sequences of complete metrizable Abelian topological groups.

LEMMA 5.13. *Let $\{\mathcal{X}_n\}_{n\geqslant 0}$ be a sequence of $\mathcal{F}$-spaces and for each $m \geqslant n \geqslant 0$ let $T_{n,m} : \mathcal{X}_m \to \mathcal{X}_n$ be a continuous operator with dense range satisfying that $T_{n,n}$ is the identity operator and $T_{k,n}T_{n,m} = T_{k,m}$ for $m \geqslant n \geqslant k$. Then $\bigcap_{n=0}^\infty T_{0,n}(\mathcal{X}_n)$ is dense in $\mathcal{X}_0$.*

THEOREM 5.14. *Let φ be a continuous self-map of $[0,1)$ with $\varphi(0) = 0$. Then the following are equivalent.*

(a) *The operator V_φ acting on $\mathcal{C}_0[0,1)$ is weakly hypercyclic.*

(b) *The operator V_φ acting on $\mathcal{C}_0[0,1)$ is hypercyclic.*

(c) *The map φ is strictly increasing and $\varphi(x) > x$ for $0 < x < 1$.*

Proof. Clearly, (b) implies (a). According to Proposition 5.12, (a) implies (c). It remains to show that (c) implies (b). Thus, suppose that (c) is satisfied. Let E be the space of bounded functions of $\mathcal{C}_0[0,1)$, which is clearly dense in $\mathcal{C}_0[0,1)$. Since V_φ acting on $L^\infty[0,1]$ has norm strictly less than 1, we see that $\|V_\varphi^n f\|_\infty$ tends to 0 as n tends to ∞. Hence, $V_\varphi^n f$ tends to 0 in $\mathcal{C}_0[0,1)$ for each f in E.

Since (c) implies that $\varphi(x)$ tends to 1 as x tends to 1, φ extends to a one-to-one continuous map of $[0,1]$ onto itself. Take $0 < a_0 < 1$ and set $a_n = \varphi_n(a_0)$ for each n in $\mathbb{Z}$. Then $\{a_n\}$ tends to 1 as n tends to ∞ and $\{a_n\}$ tends to 0 as n tends to $-\infty$. Consider also the decreasing bilateral sequence of closed subspaces of $\mathcal{C}_0[0,1)$ defined by $G_n = \{f \in \mathcal{C}_0[0,1) \text{ such that } f|_{[0,a_n]} = 0\}, n \in \mathbb{Z}$. It is elementary to check that $V_\varphi(G_{n+1})$ is a dense subspace of G_n and $V_\varphi^{-1}(G_n) = G_{n+1}$ for each n in $\mathbb{Z}$. Let k be any integer. Applying Lemma 5.13 to $\mathcal{X}_n = G_{k+n}$ and $T_{n,m} : \mathcal{X}_m \to \mathcal{X}_n$, $m \geqslant n$, defined by the restriction of V_φ^{m-n} to $\mathcal{X}_m = G_{k+m}$, we obtain $\bigcap_{n=0}^\infty V_\varphi^n(G_{k+n})$ is dense in G_k for each $k \in \mathbb{Z}$. Since $G = \bigcup_{n\in\mathbb{Z}} G_k$ is dense in $\mathcal{C}_0[0,1)$, it follows that $F = G \cap \bigcap_{n=0}^\infty V_\varphi^n(\mathcal{C}_0[0,1))$ is dense in $\mathcal{C}_0[0,1)$.

Now, since F is contained in the range of V_φ, for each f in F there is a unique Sf in $\mathcal{C}_0[0,1)$ such that $V_\varphi Sf = f$. We also have $V_\varphi^{-1}(G) = G$ because $V_\varphi^{-1}(G_n) = G_{n+1}$ for each n in Z and, therefore, $V_\varphi^{-1}(F) = F$. Hence S is well defined from F into itself. Using that $V_\varphi^{-1}(G_n) = G_{n+1}$ once again, we see that $\inf \operatorname{supp} S^n f$ tends to 1 as n tends to ∞ for each f in F. Hence $S^n f$ tends to 0 as n tends to ∞ for each f in F. Applying Corollary 4.13 with $T_n = V_\varphi^n$ and $S_n = S^n$ and with $\lambda_k = 1$, we obtain that V_φ is hypercyclic. □

Theorem 5.14 implies that $\varphi(x) < x$ for $0 < x < 1$ is not possible whenever V_φ is hypercyclic on $\mathcal{C}_0[0,1)$. This is not true if we just consider supercyclicity.

Proposition 5.15. *Let φ be a strictly continuous self-map of $[0,1)$ with $\varphi(x) < x$ for $0 < x < 1$. Then V_φ acting on $\mathcal{C}_0[0,1)$ is supercyclic.*

Proof. Let $\{a_n\}$ be a strictly increasing sequence of positive numbers such that $\{a_n\}$ tends to 1 as n tends to ∞ and let P_n from $\mathcal{C}_0[0,1)$ onto $\mathcal{C}_0[0,a_n]$ be the projections defined by $P_n f = f|_{[0,a_n]}$. Since $\varphi(x) < x$ for $0 < x < 1$, we obtain that $P_n V_\varphi^k f = V_{\psi_n}^k P_n f$ for each f in $\mathcal{C}_0[0,1)$, where $\psi_n = \varphi|_{[0,a_n]}$ and V_{ψ_n} acts on $\mathcal{C}[0,a_n]$. By Theorem 5.9 the operators V_{β_n}, where $\beta_n(x) = a_n^{-1}\psi_n(a_n x)$, acting on $\mathcal{C}_0[0,1]$ are all supercyclic. Since a change of variables provides a similarity between V_{ψ_n} acting on $\mathcal{C}_0[0,a_n]$ and V_{β_n} acting on $\mathcal{C}_0[0,1]$, we see that each V_{ψ_n} is also supercyclic.

For each $n \geqslant 0$, there is a dense G_δ-set M_n in $\mathcal{C}_0[0,a_n]$ such that each f in M_n is supercyclic for V_{ψ_n}. Since P_n is continuous onto operator, we see that $W_n = P_n^{-1}(M_n)$ is a dense G_δ-set in $\mathcal{C}_0[0,1)$. Thus, by Baire's theorem, $W = \bigcap_{n=0}^\infty W_n$ is a dense G_δ-set in $\mathcal{C}_0[0,1)$. The fact that each f in W is supercyclic for V_φ is straightforward. The proof is complete. □

The only case not covered by Lemma 5.12, Theorem 5.14 and Proposition 5.15 is the one for which φ is strictly increasing with $\varphi(x) \leqslant x$ for each $0 \leqslant x < 1$ and there is $0 < a < 1$ for which $\varphi(a) = a$.

We close by showing a result that singles out the Volterra operator from a natural one-parametric family of Volterra composition operators.

Corollary 5.16. *Assume that $\varphi_\alpha(x) = x^\alpha$ for $0 \leqslant x \leqslant \infty$, where $0 < \alpha < 1$. Then V_{φ_α} acting on $\mathcal{C}_0[0,1)$ is hypercyclic for $\alpha < 1$, supercyclic for $\alpha > 1$ and not weakly supercyclic for $\alpha = 1$.*

Proof. The statements for $\alpha < 1$ and $\alpha > 1$ follow from Propositions 5.15 and Theorem 5.14. It remains to show that the Volterra operator $V = V_{\varphi_1}$ is not weakly supercyclic on $\mathcal{C}_0[0,1)$.

Suppose that it is weakly supercyclic. As in the proof of Proposition 5.12, we can see that V must be weakly supercyclic on $\mathcal{C}_0[0,a]$ for each $0 < a < 1$, which is not the case, as shown in [**17**]. □

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